Tutorial: Computational Methods for Aeroacoustics

Sheryl M. Grace
Dept. of Aerospace and Mechanical Engineering
Boston University
sgrace@bu.edu

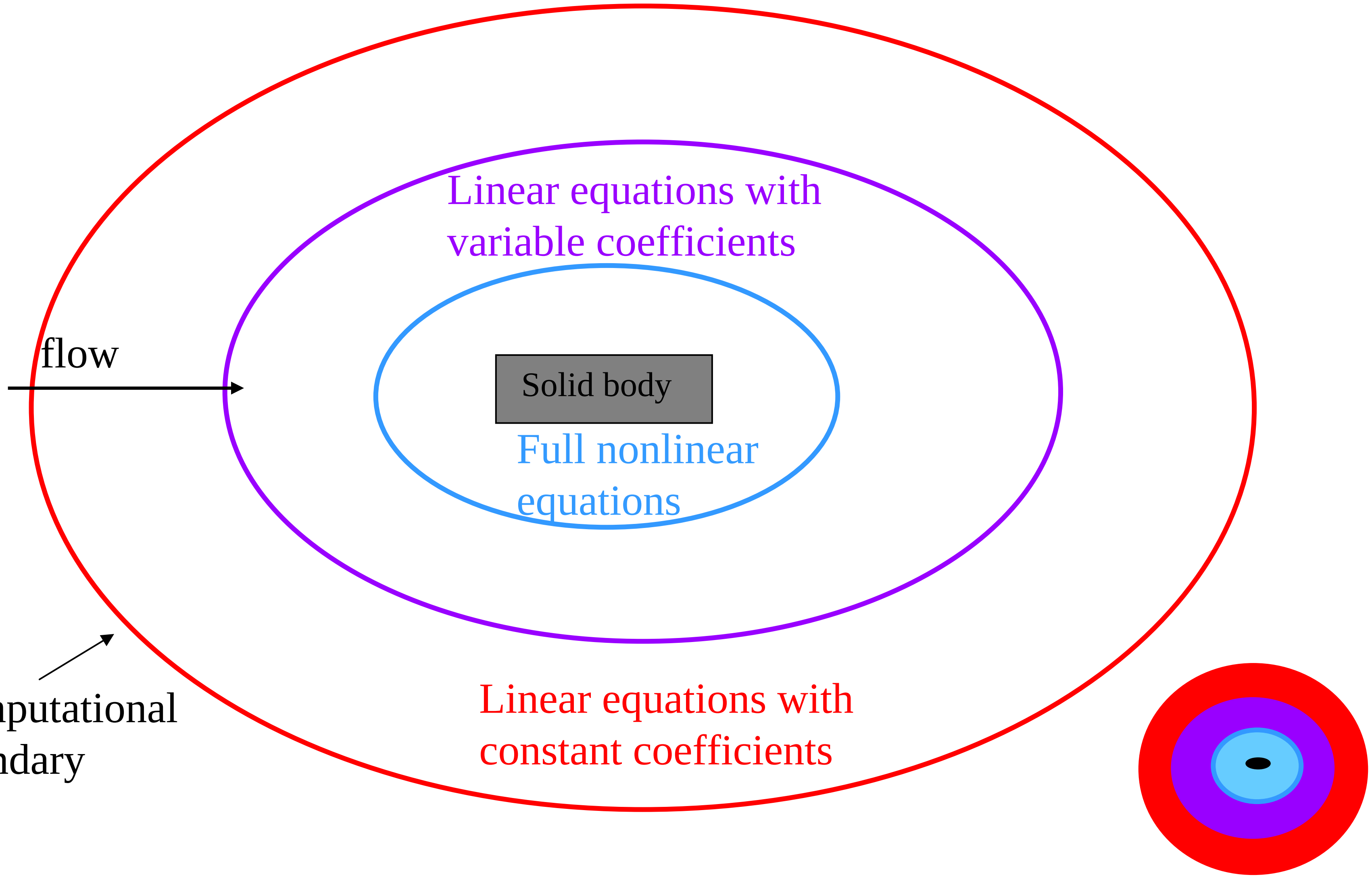
Opening comments

- I refer to ANY computational method focussing on the computation of the sound associated with a fluid flow as *computational aeroacoustics* - (CAA).
- The CAA methods are strongly linked to CFD
- CAA methods use specific techniques to resolve wave behavior well which makes this different than general computational fluid dynamics (CFD).

Kinds of applications

- Rotors
  - Helicopter noise, wind turbine noise
- Propulsion systems
  - Stator/rotor, jet noise, combustion noise, propellers (underwater)
- Airframes
  - Cavity noise, high lift wings
- Automobiles
  - Cavities, mirrors
- HVAC and piping systems
  - Fans, duct acoustics

Aeroacoustic domains



Linear equations with variable coefficients

flow

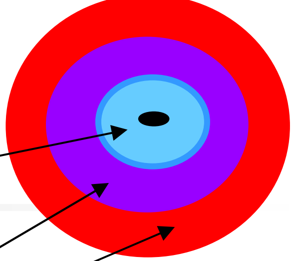
Solid body

Full nonlinear equations

Computational boundary

Linear equations with constant coefficients

Main approaches being applied



- Direct numerical, large eddy, and detached eddy simulationsDNS/LES/DES
  - Useful for problems where the sound is from turbulence
- Euler and Linearized Euler Equations (LEE)
  - LEE very popular when viscous effects can be considered 2nd order as a source of sound
- Splitting methods
  - Based on LEE, applied to specific unsteady fluid-stucture interaction problems.
- Integral approaches -- need input from something (all above + CFD)
  - Near field computation using some method above, acoustic field computed using an appropriate form of Ffowcs-Williams and Hawkings or the Kirchhoff method
- Other acoustic propagation methods
  - Solve a wave equation or associated Euler equation numerically

Outline for today

- Governing equations for the different approaches
    - Application references (at end of talk)
- Implementation of methods for CAA applied to LEE and other CFD like methods
    - ★ Dispersion relation preserving method, Padé methods
    - Boundary conditions
- Integral approaches
    - Ffowcs-Williams and Hawkings, Kirrchoff method
- Other propagators

DNS

Two - dimensional governing equations in conservative form Cartesian co-ordinate system for a perfect gas

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{E}_e}{\partial x_1} + \frac{\partial \mathbf{F}_e}{\partial x_2} - \frac{\partial \mathbf{E}_v}{\partial x_1} - \frac{\partial \mathbf{F}_v}{\partial x_2} = \mathbf{0}$$

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho e_T \end{bmatrix} \quad \mathbf{E}_e = \begin{bmatrix} \rho u_1 \\ p + \rho u_1^2 \\ \rho u_1 u_2 \\ u_1(\rho e_T + p) \end{bmatrix} \quad \mathbf{F}_e = \begin{bmatrix} \rho u_2 \\ \rho u_2 u_1 \\ p + \rho u_2^2 \\ u_2(\rho e_T + p) \end{bmatrix} \quad \mathbf{E}_v = \begin{bmatrix} 0 \\ \tau_{11} \\ \tau_{12} \\ u_1\tau_{11} + u_2\tau_{12} - q_1 \end{bmatrix} \quad \mathbf{F}_v = \begin{bmatrix} 0 \\ \tau_{21} \\ \tau_{22} \\ u_1\tau_{21} + u_2\tau_{22} - q_2 \end{bmatrix}$$

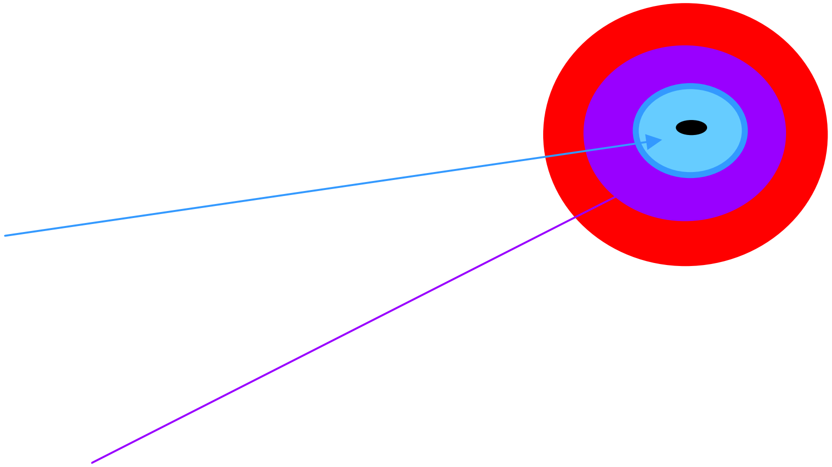
ρ - density
p - pressure
u - velocity vector
e_T - total energy
τ - viscous stress tensor
T - temperature
R - gas constant
σ - Prandtl number
γ - ratio of specific heats
q - thermal conduction

$$e_T = e + \frac{u_1^2 + u_2^2}{2} \qquad e = \frac{p}{\rho(\gamma - 1)} \qquad p = \rho R T$$

$$\tau_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_k}{\partial x_k}\right)$$

$$q_i = -\frac{\mu c_p}{\sigma}\frac{\partial T}{\partial x_i}$$

Governing equations



- DNS
- LES
- DES
- CFD
- Euler
- Linearized Euler
- Splitting method (at end)

Euler Equation

Two - dimensional governing equations in conservative form Cartesian co-ordinate system for a perfect gas

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{E}_e}{\partial x_1} + \frac{\partial \mathbf{F}_e}{\partial x_2} = \mathbf{0}$$

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho u_1 \\ \rho u_2 \\ \rho e_T \end{bmatrix} \qquad \mathbf{E}_e = \begin{bmatrix} \rho u_1 \\ p + \rho u_1^2 \\ \rho u_1 u_2 \\ u_1(\rho e_T + p) \end{bmatrix} \qquad \mathbf{F}_e = \begin{bmatrix} \rho u_2 \\ \rho u_2 u_1 \\ p + \rho u_2^2 \\ u_2(\rho e_T + p) \end{bmatrix}$$

$$e_T = e + \frac{u_1^2 + u_2^2}{2} \qquad e = \frac{p}{\rho(\gamma - 1)}$$

Moving to the conservative form of the energy equation, and noticing that the terms in the energy equation that involve can be replaced by $-\boldsymbol{u}\bullet\nabla p$, one derives a form of the equation that is useful when defining the linearized Euler equations: $\frac{Dp}{Dt} + \gamma p \bigtriangledown \cdot \mathbf{u} = 0$

LEE

Two - dimensional governing equations in conservative form
Cartesian co-ordinate system

$$\rho = \bar{\rho} + \rho'$$

$$\mathbf{u} = \bar{\mathbf{u}} + \mathbf{u}'$$

$$p = \bar{p} + p'$$

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{E}}{\partial x_1} + \frac{\partial \mathbf{F}}{\partial x_2} + \mathbf{H} = \mathbf{S}$$

S is comprised of mass, momentum, and heat sources.

$$\mathbf{U} = \begin{bmatrix} \rho' \\ \bar{\rho}u_1' \\ \bar{\rho}u_2' \\ p' \end{bmatrix} \qquad \mathbf{E} = \begin{bmatrix} \rho'\bar{u}_1 + \bar{\rho}u_1' \\ p' + \bar{\rho}\bar{u}_1 u_1' \\ \bar{\rho}\bar{u}_1 u_2' \\ \bar{u}_1 p' + \gamma\bar{p}u_1' \end{bmatrix} \qquad \mathbf{F} = \begin{bmatrix} \rho'\bar{u}_2 + \bar{\rho}u_2' \\ \bar{\rho}\bar{u}_2 u_1' \\ p' + \bar{\rho}\bar{u}_2 u_2' \\ \bar{u}_2 p' + \gamma\bar{p}u_2' \end{bmatrix}$$

$$\mathbf{H} = \begin{bmatrix} 0 \\ (\bar{\rho}u_1' + \rho'\bar{u}_1)\dfrac{\partial \bar{u}_1}{\partial x_1} + (\bar{\rho}u_2' + \rho'\bar{u}_2)\dfrac{\partial \bar{u}_1}{\partial x_2} \\ (\bar{\rho}u_1' + \rho'\bar{u}_1)\dfrac{\partial \bar{u}_2}{\partial x_1} + (\bar{\rho}u_2' + \rho'\bar{u}_2)\dfrac{\partial \bar{u}_2}{\partial x_2} \\ (\gamma - 1)p' \nabla \cdot \bar{\mathbf{u}} - (\gamma - 1)\mathbf{u}' \cdot \nabla \bar{p} \end{bmatrix}$$

If the mean flow is uniform, $\mathbf{H} = \mathbf{0}$.

Solution methods for aeroacoustics

- DRP (spatial)
- Pade (spatial)
- Time marching
- Boundary conditions

Why the need for special schemes?

A numerical representation of a PDE gives rise to an approximate solution.

❖ A consistent, stable, and convergent high order scheme does **not** guarantee good numerical wave solutions.

❖ Vortical and entropic waves are nondispersive, nondissipative, and directional

❖ Acoustic waves are nondispersive, nondissipative, and propagate with s.o.s.

❖ All CFD techniques have some dissipation (many add artificial dissipation) to stabilize the schemes and the schemes are dispersive.

(dissipation: amplitude of error)
(dispersion: phase of error)

Comments and more detail found in Tam&Web,
J. of Comp. Phys 107:262-281,1993.

Dispersion-Relation Preserving Schemes (DRP)

The number of wave modes and their wave propagation characteristics is found through the dispersion relation.

Dispersion relation: functional relation between the ω and k (angular frequency and wave number)

The dispersion relation is found by considering the space and time Fourier transforms of the governing equations. The LEE can be transformed into a simple matrix system:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{E}}{\partial x_1} + \frac{\partial \mathbf{F}}{\partial x_2} + \mathbf{H} = \mathbf{S}$$

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{C}_1 \frac{\partial \mathbf{U}}{\partial x} + \mathbf{C}_2 \frac{\partial \mathbf{U}}{\partial y} = G$$

$$\mathbf{U} = \begin{bmatrix} \rho' \\ \bar{\rho} u_1' \\ \bar{\rho} u_2' \\ p' \end{bmatrix}$$

$$\mathbf{A}\hat{\mathbf{U}} = \hat{\mathbf{G}}$$

The matrix **A** has 3 distinct eigenvalues. The repeated eigenvalue is associated with an eigenvector describing the entropy wave field, and an eigenvector describing the vortical wave field. The other 2 eigenvalues are associated with the acoustic wave field.

DRP scheme (Tam,Web, J. Comp. Phys. 107:262-281,1993)

If one wants to match the dispersion relation in the numerical simulation, then one must match the Fourier transform.

The finite difference representation of a derivative takes the form:

$$\frac{\partial f}{\partial x}(x) \simeq \frac{1}{\Delta x} \sum_{j=-N}^{M} a_j f(x + j \triangle x)$$

Defining the Fourier transform as $\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$

The transform of the finite difference expression becomes

$$ik\hat{f} \simeq \left(\frac{1}{\Delta x} \sum_{j=-N}^{M} a_j e^{ik(j\Delta x)} \right) \hat{f}$$

So we see the approximation to the wave number is $\tilde{k} \Delta x = -i \sum_{j=-N}^{M} a_j e^{ijk\Delta x}$

DRP schemes formed by minimizing the wave number error, i.e. minimizing

$$E = \int_{-\pi/2}^{\pi/2} |k \triangle x - \tilde{k} \Delta x|^2 d(k \triangle x)$$

DRP scheme cont.

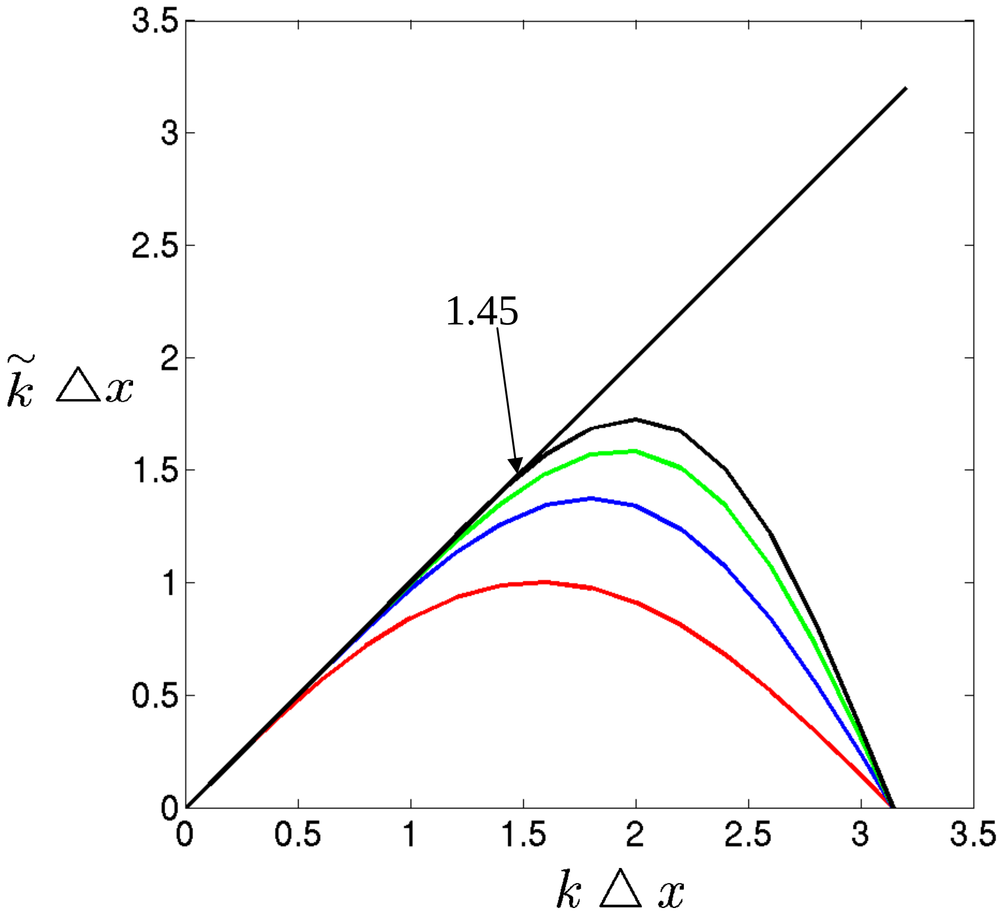
For symmetric stencils (central difference type), $\tilde{k}$ is a real number. (For nonsymmetric stencils, one gets complex values and often spatially growing wave solutions.)

Combination of the Taylor series finite difference approximation for **N=M=3**, and the minimization function, gives a **fourth** order accurate discretization

Scheme	j=-3	j=-2	j=-1	j=0	j=1	j=2	j=3	order
DRP	-0.02651995	0.18941314	-0.79926643	0	0.79926643	-0.18941314	0.02651995	4th
Central	0	0	-1/2	0	1/2	0	0	2nd
Central	0	1/12	-2/3	0	2/3	-1/12	0	4th
Central	-1/60	3/20	-3/4	0	3/4	-3/20	1/60	6th

Comparison of wave number accuracy

$$\tilde{k}\,\triangle x = -i \sum_{j=-N}^{M} a_j e^{ijk\triangle x}$$



$$k = \frac{2\pi}{\lambda}$$

$$\lambda = \frac{2\pi}{1.45} = 4.33$$

Waves with wavelengths longer than 4.33 grid spacings will be adequately approximated

DRP

6th order Central

4th order Central

2nd order Central

Padé/compact methods (Lele, J. Comp. Phys. 103:16-42,1992)

Match Taylor's series coefficients of various orders given:

$$a_2 f'_{j-2} + a_1 f'_{j-1} + f'_j + a_1 f'_{j+1} + a_2 f'_{j+2} = b_3 \frac{f_{p+3} - f_{p-3}}{6 \triangle x} + b_2 \frac{f_{p+2} - f_{p-2}}{4 \triangle x} + b_1 \frac{f_{p+1} - f_{p-1}}{2 \triangle x}$$

Fourier transform of the space variable gives

$$i\,\tilde{k}\,\hat{f}\left(1 + \sum_{j=1}^{2} a_j\left(e^{ik\Delta xj} + e^{-ik\Delta xj}\right)\right) = \hat{f}\left(\sum_{j=1}^{3} \frac{b_j}{2 \triangle xj}\left(e^{ik\Delta xj} - e^{-ik\Delta xj}\right)\right)$$

So the approximate wave number is given by $\tilde{k} \triangle x = \dfrac{\sum_{j=1}^{3} \frac{b_j}{j} \sin(k \triangle xj)}{1 + \sum_{j=1}^{2} 2a_j \cos(k \triangle xj)}$

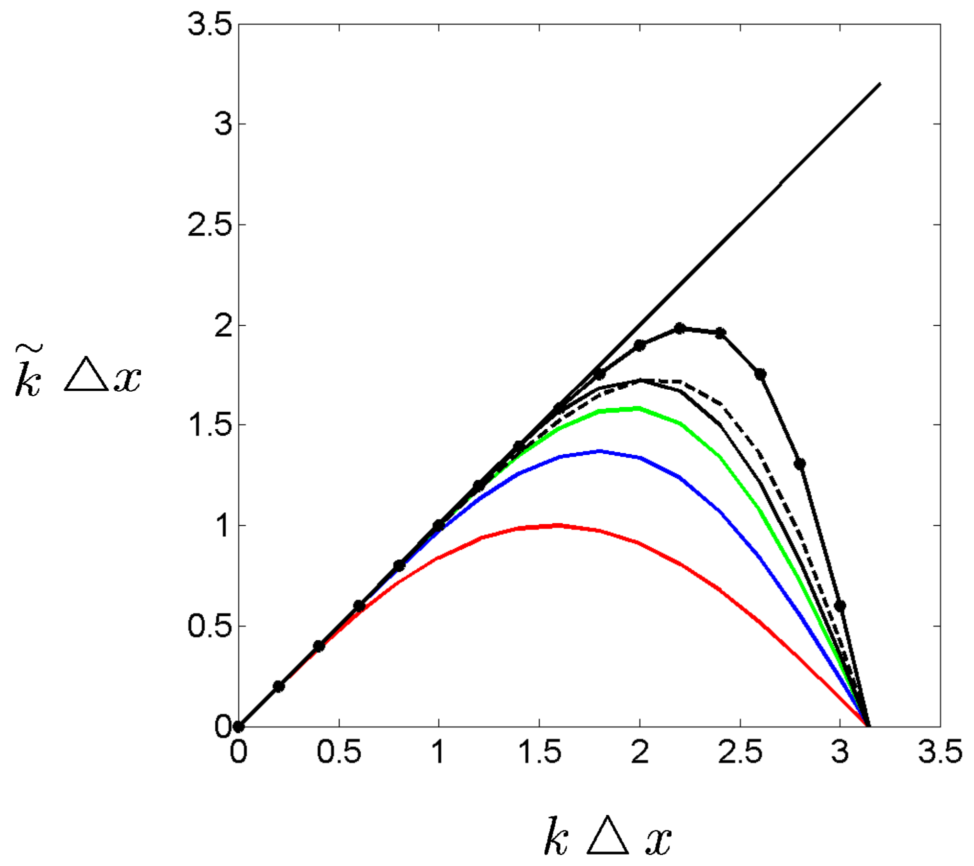
Tridiagonal schemes, 3 pt. stencil l.h.s., 5 pt. stencil r.h.s

Scheme	a1	a2	b1	b2	b3			order
Padé	0.25	0	0.6666666	0	0			4th
Padé	0.33333333	0	4.6666666	0.11111111	0			6th

Comparison of wave number accuracy

Comment by Wolfgang Schroder : VKI lecture 04

2D: DRP $\sqrt{3}$ faster than Padé
3D: DRP an order faster



Padé (6th order) (*)
Padé (4th order) (dashed)
DRP (4th order)
6th order Central
4th order Central
2nd order Central

Time discretization

One can use a time discretization scheme constructed using the DRP methodology. (Tam and Web)
The method requires the addition of some artificial damping because of the existence of spurious short waves.

Tam and Dong: J. of Comp. Acoustics, 1:1-30, 1993;
Add artificial damping such that the damping is confined to the high wavenumber range.
Then the short waves are damped leaving the long waves basically unaffected.

Zhuang and Chen: AIAA J. 40(3):443-449, 2002;
Use high-order optimized upwind schemes that damp out the spurious short waves automatically.

➡ Hu et al, J, of Comp. Phys, 124:177-191, 1996;
Use low-dissipation low-dispersion, low-storage Runge-Kutta schemes.

LDDRK (Hu)

Consider the time discretization

$$\frac{\partial \mathbf{U}}{\partial t} = F(\mathbf{U})$$

$$\mathbf{K}_i = \triangle t \;\; F(\mathbf{U}^n + \beta_i \mathbf{K}_{i-1})$$

$$\mathbf{U}^{n+1} = \mathbf{U}^n + \mathbf{K}_p$$

applied to the one-dimensional wave equation

$$\frac{\partial f}{\partial t} + c\frac{\partial f}{\partial x} = 0$$

where from the spatial transform discussed earlier

$$\frac{\partial \hat{f}}{\partial t} + ic\,\tilde{k}\,\hat{f} = 0.$$

so that $F(\mathbf{U}) = (-ic\,\tilde{k}\;\mathbf{U})$

$$\hat{\mathbf{U}}_k^{n+1} = \hat{\mathbf{U}}_k^n \left(1 + \sum_{m=1}^{p} \gamma_m (-ic\,\tilde{k}\;\;\triangle t)^m\right)$$

$$\begin{aligned} \gamma_1 &= 0 \\ \gamma_2 &= \beta_p \\ \gamma_3 &= \beta_p \beta_{p-1} \\ \gamma_p &= \beta_p \beta_{p-1} \ldots \beta_2 \end{aligned}$$

Define the amplification factor for the scheme

$$r = \frac{\hat{\mathbf{U}}_k^{n+1}}{\hat{\mathbf{U}}_k^n} = 1 + \sum_{m=1}^{p} \gamma_m (-i\sigma)^m$$

LDDRK cont.

$$r = \frac{\hat{\mathbf{U}}_k^{n+1}}{\hat{\mathbf{U}}_k^{n}} = 1 + \sum_{m=1}^{p} \gamma_m (-i\sigma)^m$$

$$r_{exact} = e^{-ic\tilde{k}\Delta t} = e^{-i\sigma}$$

$$\frac{r}{r_{exact}} = |r| e^{-i\delta}$$

Dissipation error

Dispersion error

Classical Rung-Kutta matches the expansion for $e^{-i\sigma}$

So that $\gamma_1 = 1$, $\gamma_2 = 1/2!$, $\gamma_3 = 1/3!$, $\gamma_4 = 1/4!$ would give a fourth order approximation

Specifies range of σ

Here γ_m chosen to minimize $$E = \int_0^{\Gamma} \left|1 + \sum_{m=1}^{p} \gamma_m (-i\sigma)^m - e^{-i\sigma}\right|^2 d\sigma$$

and to satisfy the stability limit |r| <= 1

LDDRK (cont.)

$$\hat{\mathbf{U}}_k^{n+1} = \hat{\mathbf{U}}_k^n \left(1 + \sum_{m=1}^{p} \gamma_m (-ic\,\tilde{k}\,\Delta t)^m\right)$$

For all schemes: $\gamma_1 = 1$, $\gamma_2 = 1/2!$

Stages	γ_3	γ_4	γ_5	γ_6	L	R		Order
4	0.1629978	0.0407574	-	-	0.085	2.85		2nd
5	0.166558	0.0395041	0.00781071	-	1.35	3.54		2nd
6	1/3!	1/4!	0.00781005	0.00132141	1.75	1.75		4th

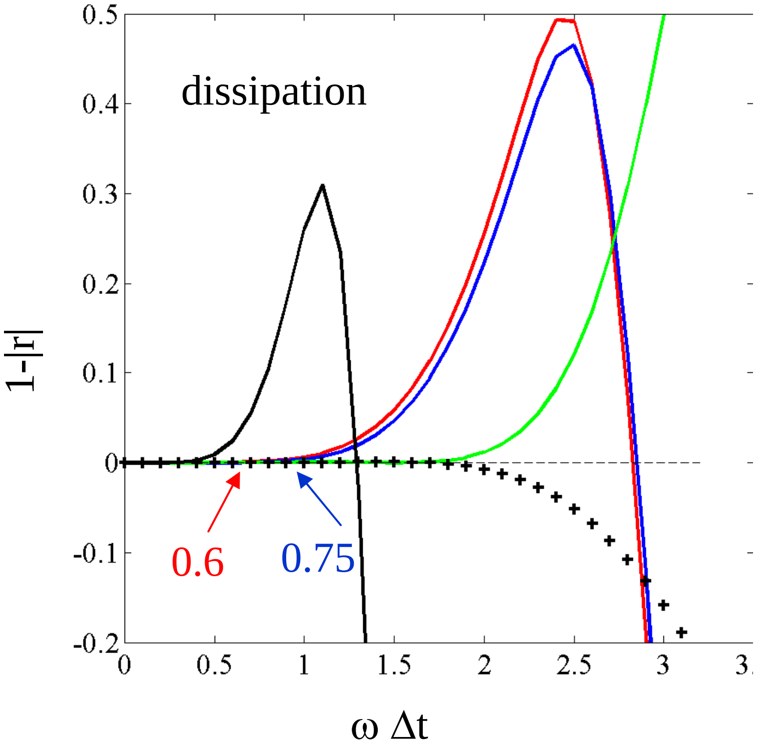
accuracy limit ↑ L

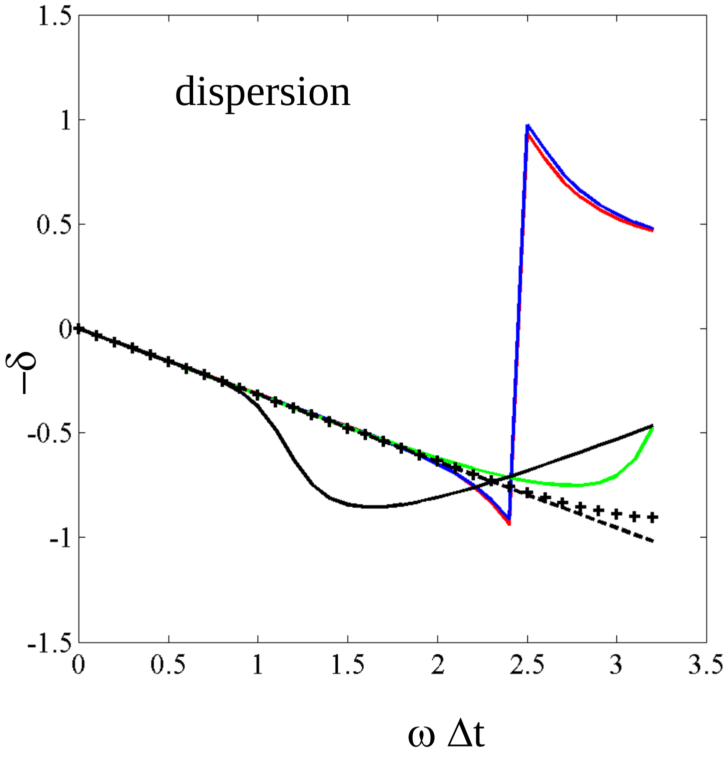
stability limit ↑ R

Hu et al. discusses the equivalence of this method to minimizing the error in the dispersion error. Also, the implementation of boundary conditions is discussed.

Bogey, Bailly, J. of Comp. Phys, 194:194-219, 2004 is another good example of applying this method to define a LDDRK.

Time discretization comparison





4th order 6 stage LDDRK (*)
3rd order DRP
2nd order 5 stage LDDRK
2nd order 4 stage LDDRK
2nd order 4stage classical RK
Exact (dashed)

Boundary conditions

One must set computational boundaries that draw a line between a flow region of interest and other regions that are to be neglected.

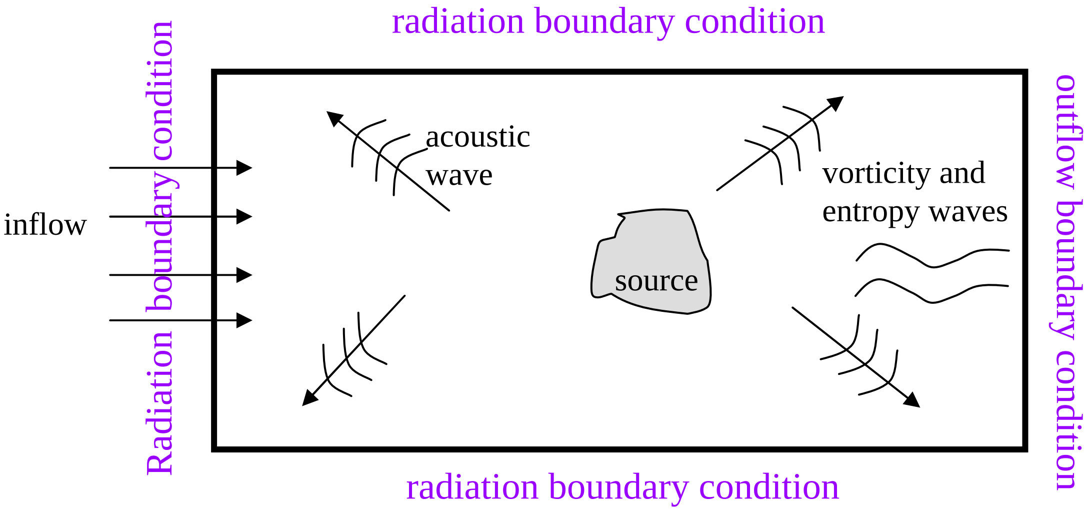
These boundaries must not produce unrealistic reflections or spurious solutions

BC type	Pro	Con
Characteristic-based	Straightforward and robust	Inaccurate for wave angles not perpendicular to the boundary
Asymptotic **	Accurate when applicable	BC surface must be in far-field, not always applicable
Buffer zone	Quite familiar to CFD community	Large zone, may produce some reflections at interface
Perfectly Matched Layer **	Absorbs well with smaller zone, no reflections	Stability has been an issue, seems to be better now. Set up for linearized eqs.

BC’s cont.

Consider the asymptotic boundary conditions (and BC’s that will apply be applied in the linearized region of the flow field)

For inviscid, nonheat-conducting, calorically perfect gas, one can decouple the equations into equations that govern the vorticity fluctuation, the entropy fluctuation, and the pressure fluctuations.



Good reviews:

Colonius, Annu. Rev. Fluid Mech. 36:315-45, 2004
Givoli, J. Comp. Phys, 94(1):1-29, 1991
Hagstrom , Acta. Numerica, 8:47-106, 1999

Asymptotic BC's

Tam and Web : LEE, uniform flow in x-direction only, simple starting point.

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{E}}{\partial x_1}+\frac{\partial \mathbf{F}}{\partial x_2}+\mathbf{H}=\mathbf{S} \qquad \frac{\partial \mathbf{U}}{\partial t}+\mathbf{C}_1\frac{\partial \mathbf{U}}{\partial x}+\mathbf{C}_2\frac{\partial \mathbf{U}}{\partial y}=G \qquad \mathbf{U}=\begin{bmatrix}\rho' \\ \bar{\rho}u_1' \\ \bar{\rho}u_2' \\ p'\end{bmatrix}$$

Transformed system $\mathbf{A}\hat{\mathbf{U}}=\hat{\mathbf{G}}$

Already mentioned that **A** has 3 distinct eigenvalues giving rise to 4 eigenvectors, $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, $\mathbf{X}_4$ where $\mathbf{X}_1$ can be associated with the entropy wave, $\mathbf{X}_2$ can be associateed with the vorticy wave, and $\mathbf{X}_3$ & $\mathbf{X}_4$ can be associated with the two modes of the acoustic waves.

Radiation BC (Tam/Web)

$\mathbf{A}\hat{\mathbf{U}} = \hat{\mathbf{G}}$ can be solved in terms of the eigenvectors: $\hat{\mathbf{U}} = \sum_{j=1}^{4} \frac{B_j}{\lambda_j}\mathbf{X}_j \quad \mathbf{B} = [\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4]^{-1}\hat{\mathbf{G}}$

Focus on the two acoustic modes, in particular, the outgoing mode at a boundary. Transform back to space and time using

$$\mathbf{U} = \int_\Gamma \int \int_{-\infty}^{\infty} \left[\frac{B_{3,4}}{\lambda_{3,4}}\mathbf{X}_{3,4}\right] e^{i(\alpha x+\beta y-\omega t)}\ d\alpha\ d\beta\ d\omega$$

denotes acoustic part

You get a solution in the form of

$$\begin{bmatrix}\rho'\\ u'\\ v'\\ p'\end{bmatrix} = \begin{bmatrix}\rho_a\\ u_a\\ v_a\\ p_a\end{bmatrix} = \frac{F(r/V(\theta)-t,\theta)}{r^{1/2}}\begin{bmatrix}\frac{1}{\bar{c}^2}\\ \frac{\hat{u}(\theta)}{\overline{\rho c}}\\ \frac{\hat{v}(\theta)}{\overline{\rho c}}\\ 1\end{bmatrix} + O(r^{-3/2})$$

V velocity of propagation in θ direction $V(\theta) = \bar{c}[M\cos\theta + (1-M^2\sin^2\theta)^{1/2}]$

In general this states that the acoustic disturbance satisfies:

$$\left(\frac{1}{V(\theta)}\frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{1}{2r}\right)\begin{bmatrix}\rho'\\ u'\\ v'\\ p'\end{bmatrix} = 0 + O(r^{-5/2})$$

Radiation (asymptotic) BC

$$\left(\frac{1}{V(\theta)}\frac{\partial}{\partial t}+\frac{\partial}{\partial r}+\frac{1}{2r}\right)\begin{bmatrix}\rho'\\ u'\\ v'\\ p'\end{bmatrix}=0+O(r^{-5/2})$$

$$V(\theta)=\bar{c}[M\cos\theta+(1-M^2\sin^2\theta)^{1/2}]$$

Other conditions that are similar

Bayliss, Turkel, J. Comp. Phys. 48:182-199, 1982

Hagstrom, Hariharan, SIAM J. Sci. Comput., 25(3):1088:1101, 2003 (high order!)

Outflow (asymptotic) BC (Tam/Web, 1-D uniform mean flow)

At the outflow, entropy, vorticity, and acoustic waves must traverse the boundary.

Use the same process of evaluating the behavior of the entropic and vortical parts of the solution. The density perturbation is associated with the entropic mode and the vortical mode is associated with the perturbation velocity vector.

The form of **U** at the outflow is then

$$\begin{bmatrix} \rho' \\ u' \\ v' \\ p' \end{bmatrix} = \begin{bmatrix} \chi_1(x-\overline{u}t, y) + \rho_a \\ \chi_2(x-\overline{u}t, y) + u_a \\ \chi_3(x-\overline{u}t, y) + v_a \\ p_a \end{bmatrix} + \dots$$

Outflow boundary equations

$$\begin{cases} \dfrac{1}{V(\theta)}\dfrac{\partial p'}{\partial t} + \cos\theta\dfrac{\partial p'}{\partial x} + \sin\theta\dfrac{\partial p'}{\partial y} + \dfrac{p'}{2r} = 0 \\ \dfrac{\partial \rho'}{\partial t} + \overline{u}\dfrac{\partial \rho'}{\partial x} = \dfrac{1}{\overline{c}^2}\left(\dfrac{\partial p'}{\partial t} + \overline{u}\dfrac{\partial p'}{\partial x}\right) \\ \dfrac{\partial u'}{\partial t} + \overline{u}\dfrac{\partial u'}{\partial x} = -\dfrac{1}{\overline{\rho}}\dfrac{\partial p'}{\partial x} \\ \dfrac{\partial v'}{\partial t} + \overline{u}\dfrac{\partial v'}{\partial x} = -\dfrac{1}{\overline{\rho}}\dfrac{\partial p'}{\partial y} \end{cases}$$

$$V(\theta) = \overline{c}[M\cos\theta + (1 - M^2\sin^2\theta)^{1/2}]$$

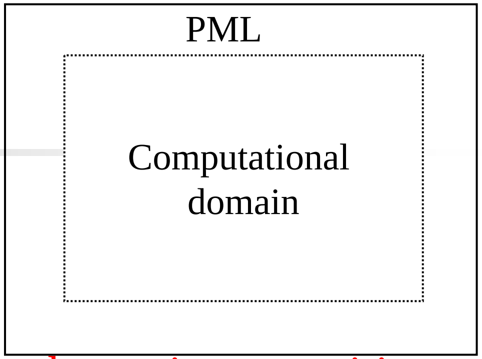
BC’s cont.

Tam and Dong: J. of Comp. Acoustics. 4(2):175-201, 1996;

Extended the work of Tam and Web to multidimensional nonuniform mean flow. The boundary condition equations have the same form excep that

$$\bar{\mathbf{u}}\frac{\partial}{\partial x} \rightarrow \bar{\mathbf{u}} \cdot \bigtriangledown$$

Perfectly matched layer



Following Hu, J. of Comp. Phys. 173:455-480, 2001.

- Add absorbing layer at end of computational domain where the absorption quantities are based on the plane wave solutions of the linearized Euler equations.
- Create the absorption quantities such that all three wave types are absorbed in the appropriate area.
- PML differs from the regular buffer zone technique in that the equations used in the added region will not cause ANY reflection when entering the region at any frequency and angle of incidence.

Hu uses the nondimensional form of LEE, but follows the same process of forming the plane wave solutions (based on the dispersion relations) discussed previously.

One forms

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{C}_1 \frac{\partial \mathbf{U}}{\partial x} + \mathbf{C}_2 \frac{\partial \mathbf{U}}{\partial y} = 0 \qquad \mathbf{U} = \begin{bmatrix} \rho' \\ \bar{\rho} u_1' \\ \bar{\rho} u_2' \\ p' \end{bmatrix}$$

$$\mathbf{A}\hat{\mathbf{U}} = \hat{\mathbf{G}}$$

but in nondimensional quantities, $\bar{\rho}$ is 1.

and then find the eigenvalues of **A** which when set to zero give the dispersion relations.

PML cont.

Combined acoustic eigenvalues give $(\omega - Mk_x)^2 - k_x^2 - k_{\mathbf{y}}^2 = 0$

$$k_x = \frac{\omega\cos\phi}{1 + M\cos\phi} \qquad k_y = \frac{\omega\sin\phi}{1 + M\cos\phi}$$

wave angle

Eigenvalue for the entropy and vortical modes gives $\omega - Mk_x = 0$

$$\underbrace{k_x = \frac{\omega}{M} \qquad k_y = \frac{\omega\tan\psi}{M}}_{\text{entropy}} \qquad\qquad \underbrace{\frac{\omega}{M} = k_x \qquad k_y = \frac{\omega\tan\chi}{M}}_{\text{vortical}}$$

A single Fourier/Laplace component of U is still formed from $\hat{\mathbf{U}} = \frac{B_j}{\lambda_j}\mathbf{X}_j e^{ik_x x + ik_y y - i\omega t}$
where **X**_j are the eigenvectors

PML cont.

Absorption coefficients are then introduced using a splitting method

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{C}_1 \frac{\partial \mathbf{U}}{\partial x} + \mathbf{C}_2 \frac{\partial \mathbf{U}}{\partial y} = 0$$

$$\mathbf{U} = \mathbf{U}_1 + \mathbf{U}_2$$

absorption coefficients

$$\frac{\partial \mathbf{U}_1}{\partial t} + \sigma_x \mathbf{U}_1 + \mathbf{C}_1 \frac{\partial \mathbf{U}_1}{\partial x} = 0 \qquad \frac{\partial \mathbf{U}_2}{\partial t} + \sigma_y \mathbf{U}_2 + \mathbf{C}_2 \frac{\partial \mathbf{U}_2}{\partial y} = 0$$

Recombining the two equations, and considering a single frequency gives

$$\boxed{-i\omega \hat{\mathbf{U}} + \frac{1}{1 + \frac{i\sigma_x}{\omega}} \mathbf{C}_1 \frac{\partial \hat{\mathbf{U}}}{\partial x} + \frac{1}{1 + \frac{i\sigma_y}{\omega}} \mathbf{C}_2 \frac{\partial \hat{\mathbf{U}}}{\partial y} = 0}$$

Clear denominator of σ and reformulate to space-time →

Introduction of scaled spatial parameters, allows one to write the acoustic mode for a given Fourier/Laplace component in such a way that the damping is clear.

$$x' = \left(1 + \frac{i\sigma_x}{\omega}\right) x \quad y' = \left(1 + \frac{i\sigma_y}{\omega}\right) y$$

$$\hat{\mathbf{U}} = \frac{B_{ac}}{\lambda_{ac}} \begin{pmatrix} 1 \\ \cos\phi \\ \sin\phi \\ 1 \end{pmatrix} \exp\left(\frac{i\omega\cos\phi}{1 + M\cos\phi} x - \frac{\sigma_x \cos\phi}{1 + M\cos\phi} x + \frac{i\omega\sin\phi}{1 + M\cos\phi} y - \frac{\sigma_y \sin\phi}{1 + M\cos\phi} y - i\omega t \right)$$

There are similar expressions for the entropy and vortical modes

PML cont.

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{C}_1 \frac{\partial \mathbf{U}}{\partial x} + \mathbf{C}_2 \frac{\partial \mathbf{U}}{\partial y} + \sigma_y \mathbf{C}_1 \frac{\partial \mathbf{q}}{\partial x} + \sigma_x \mathbf{C}_2 \frac{\partial \mathbf{q}}{\partial y} + (\sigma_x + \sigma_y)\mathbf{U} + \sigma_x \sigma_y \mathbf{U} = 0$$

$$\frac{\partial \mathbf{q}}{\partial t} = \mathbf{U}$$

It was shown that there is an instability arising due to convective acoustic waves that have a positive group velocity but a negative phase velocity in the x-direction.

Another spatial transformation is used to overcome this instability, and the final equation that one uses in the PML becomes

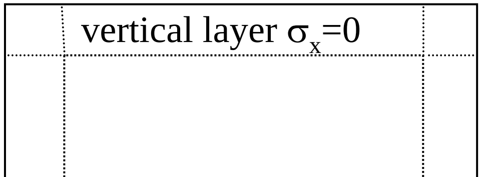
$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{C}_1 \frac{\partial \mathbf{U}}{\partial x} + \mathbf{C}_2 \frac{\partial \mathbf{U}}{\partial y} + \sigma_y \mathbf{C}_1 \frac{\partial \mathbf{q}}{\partial x} + \sigma_x \mathbf{C}_2 \frac{\partial \mathbf{q}}{\partial y} + (\sigma_x + \sigma_y)\mathbf{U} + \sigma_x \sigma_y \mathbf{U} + \frac{\sigma_x M}{1 - M^2} \mathbf{C}_1 (\mathbf{U} + \sigma_y \mathbf{q}) = 0$$

$$\frac{\partial \mathbf{q}}{\partial t} = \mathbf{U}$$

0 in vertical layer

0 in horizontal layer

q is only introduced in the PML domain.



PML cont.

The absorption coefficients can be varied gradually, for example (given in Hu's paper):

location where PML domain starts

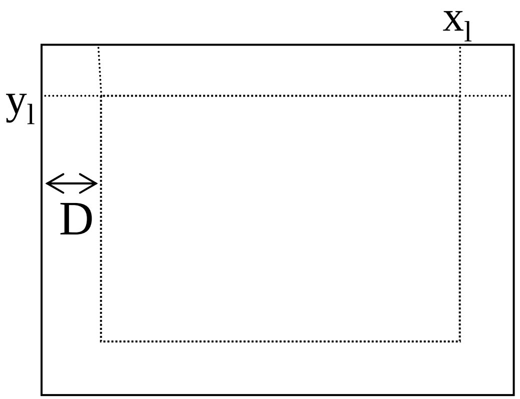
$\sigma_m \Delta x = 2$ (Δx is the grid size)
$\beta = 2$

$$\sigma_x = \sigma_m \underbrace{(1 - M^2)} \left| \frac{x - x_l}{D} \right|^{\beta} \qquad \sigma_y = \sigma_m \left| \frac{y - y_l}{D} \right|^{\beta}$$

Term to allow absorption rates to be the same in the x and y directions

width of PML domain

Outer edge of the PML can use characteristic, asymptotic, or even very simple reflective type boundary conditions.



Solution methods using the integral approach

- Acoustic analogy
- Ffowcs-Williams and Hawkings
- Kirchhoff

Flow field quantities are known in a region near the source, use the integral approaches to find the propagation of the acoustics to the far-field.

Lighthill's Eq.

creation of sound ← generation of vorticity

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)\left(c_0^2(\rho-\rho_0)\right)=\frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}.$$

refraction, convection, attenuation, known *a priori*

mean speed of sound

mean density

Lighthill stress tensor

$$\sigma_{ij}=2\mu e_{ij}-\frac{1}{3}e_{kk}\delta_{ij}$$

$$e_{ij}=\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right),$$

$$T_{ij}=\rho v_i v_j+\left((p-p_0)-c_0^2(\rho-\rho_0)\right)\delta_{ij}-\sigma_{ij}.$$

excess momentum transfer

wave amplitude nonlinearity
mean density variations

attenuation of sound

Solution to Lighthill's Eq.

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\left(c_0^2(\rho - \rho_0)\right) = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}.$$

Quadrupole like source!

Direct application of Green's function

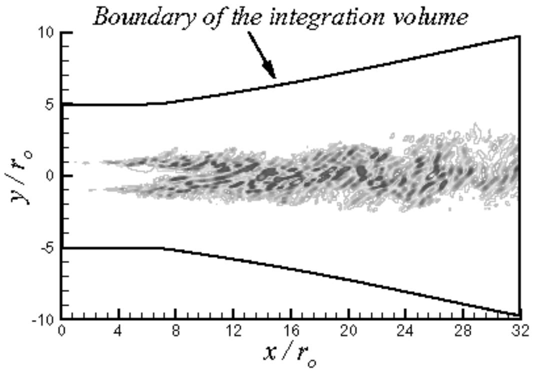
$$c_0^2(\rho - \rho_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial^2}{\partial y_i \partial y_j} T_{ij}(\mathbf{y}) \frac{\delta\left(t - \tau - \frac{|\mathbf{x}-\mathbf{y}|}{c_0}\right)}{4\pi|\mathbf{x} - \mathbf{y}|} d^3\mathbf{y}\, d\tau$$

Far-field expansion, integration with respect to retarded time

$$(\rho - \rho_0)(\mathbf{x}, t) = \frac{x_i x_j}{4\pi c_0^4 |\mathbf{x}|^3} \frac{\partial^2}{\partial t^2} \int_{-\infty}^{\infty} T_{ij}\left(\mathbf{y}, t - \frac{|\mathbf{x}|}{c_0} + \frac{\mathbf{x}\cdot\mathbf{y}}{c_0|\mathbf{x}|}\right) d^3\mathbf{y}, \quad |\mathbf{x}| \to \infty.$$

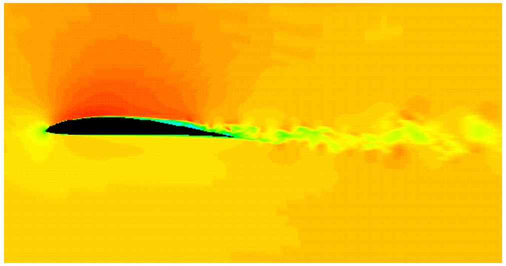
Some example applications of Lighthill’s analogy

- Uzun et al. AIAA Paper No. 2004-0517. Applied to jet flow, coupled to an LES. Source term for acoustics propagating in a specific direction (figure)



- Colonius, Freund, AIAA J. 38(2):368-370. Applied to jet flow, coupled to a DNS.

- Oberai et al. (AIAA J., 40(11):2206-2216, 2002) - airfoil self-noise, coupled to FEM LES



FWH

- Turbulence is moving
- Two distinct regions of fluid flow
- Solid boundaries in the flow

Differential form of the FWH Eq.

$$(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2)(c_0^2(\rho - \rho_0)H(f))$$

unsteady surface pressure

Reynolds stress

viscous stresses

$$= \frac{\partial^2}{\partial x_i \partial x_j}(T_{ij}H(f)) - \frac{\partial}{\partial x_i}\left((\rho v_i (v_j - V_j) + (p - p_0)\delta_{ij} - \sigma_{ij})\frac{\partial H}{\partial x_j}\right)$$

$$+\frac{\partial}{\partial t}\left((\rho(v_j - V_j) + \rho_0 V_j)\frac{\partial H}{\partial x_j}\right)$$

rate of mass transfer
across the surface

Only nonzero on the surface

FWH cont.

Integral form of the FWH Eq.

$$\begin{aligned}(\rho-\rho_0)H(f) \;&=\; \frac{\partial^2}{\partial x_i x_j}\int_{\mathcal{V}(\tau)}\frac{[T_{ij}]}{4\pi c_0^2|\mathbf{x}-\mathbf{y}|}d^3\mathbf{y} \qquad \text{Quadrupole}\\ \text{Dipole}\qquad &-\frac{\partial}{\partial x_i}\int_{S(\tau)}\left[\frac{\rho v_i(v_j-V_j)+(p-p_0)\delta_{ij}-\sigma_{ij})}{4\pi c_0^2|\mathbf{x}-\mathbf{y}|}\right]dS_j(\mathbf{y})\\ \text{Monopole}\qquad &+\frac{\partial}{\partial t}\int_{S(\tau)}\left[\frac{\rho(v_j-V_j)+\rho_0 V_j}{4\pi c_0^2|\mathbf{x}-\mathbf{y}|}\right]dS_j(\mathbf{y})\end{aligned}$$

Square brackets indicate evaluation at the retarded time

$$\tau=t-|\mathbf{x}-\mathbf{y}|/c_0$$

If *S* shrinks to the body

dipole = fluctuating surface forces

monopole = aspiration through the surface

Comments on FWH method

- There is a formulation for moving surfaces (some discussion included as an appendix of this presentation)

- There is a formulation for permeable (or porous) surfaces
  - developed for use with CFD where the surface has to be placed quite close to the body, but not on the body
  - Paper to appear in AIAA J (in near future) - A. Morgans et al. CFD + permeable surface FWH for transonic helicopter noise

- Brentner, Farrassat, Progress in Aerospace Sciences 39:83-120, 2003
  - Great review/overview of use of FWH in rotor noise studies

- Gloerfelt et al., JSV 266:119-146, 2003, FWH and porous FWH for 2D cavity problem coupled to DNS
  - Acoustic part computed in the **frequency domain** (no need for retarded time variable this way.)

- Kim at colleagues at FLUENT, AIAA Paper No. 2003-3202, couples FWH to their LES solver.

Comment on Kirchhoff method

- Solve the homogeneous wave equation using the free-space Green's function approach

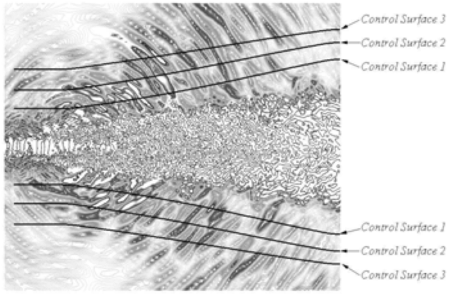
- All sources of sound and nonuniform flow regions must be inside the surface of integration. Integration surface must be placed in the linear region of the flow.

- FWH is same if the surface is chosen as it is for the Kirchhoff method

- FWH superior
    - Based on the governing equation of motion (not wave equation)
    - Valid in the nonlinear region

- Both methods when used in the linear regime, may capture a lower maximum frequency (CFD method may use grid-stretching).

Kirchhoff and FWH cont.



- Brentner, Farrassat AIAA J. 36(8):1379-1386, 1998
  - Compare FWH and Kirchhoff
  - FWH does separate contributions to the noise (if surface is placed closed to or on body)

- Patrick(Grace) et al.ASME FED 147:41-46, 1993, used the Kirchhoff method coupled to the splitting method for fluid/airfoil interaction noise

- Gloerfelt et al. also compares the two methods (for the cavity problem)

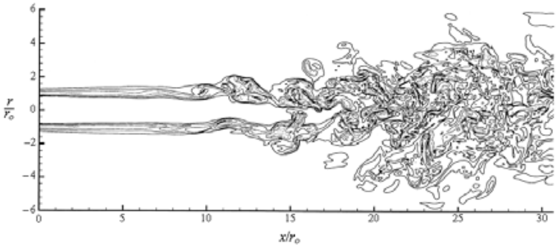
- Lyrintzis gives a great review of coupling CFD to FWH and Kirchhoff in Int. J. of Aeroacoustics 2(2):95-128, 2003

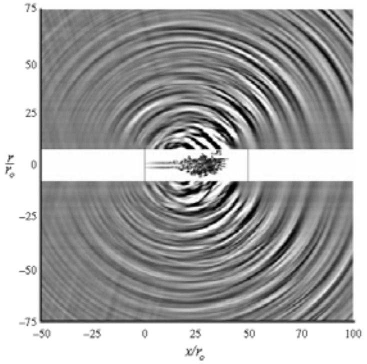
- Uzun et al., AIAA Paper No. 2004-0517, shows LES coupled to open FWH and Kirchhoff surfaces for jet (meaning jet outflow not enclosed by the surface - see figure above)

- Rahier et al. Aero. Sci. & Tech. 8:453-467,2004, **open vs. closed** surfaces for jet analysis. Open surface makes more sense.

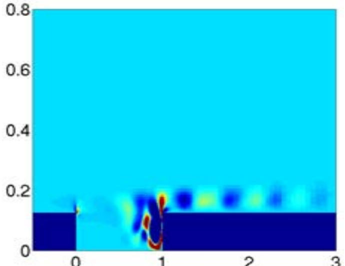
Alternative couplings

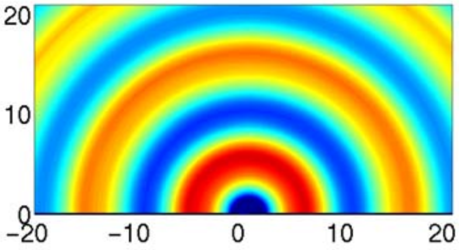
- Freund, J. of Comp. Phys. 157:796-800, 2000. Solve the linearized Euler equations (with an additional term) using near-field DNS or LES as boundary information. Additional term drives the density towards the Navier-Stokes value. Applied to jet M = 0.9, JFM 438:277-305,2001)





- Grace, Curtis. ASME NCAD, 1999. Low M applications, compute solution to wave equation in appropriate region using CFD input

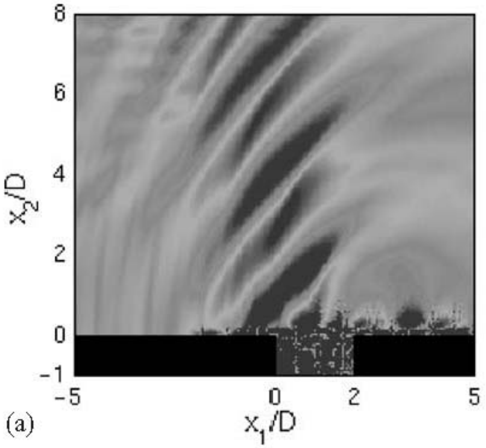




Some more CAA applications in the literature

DNS examples

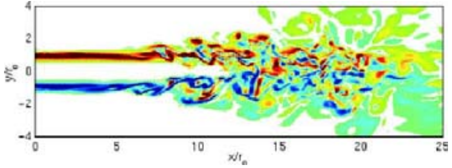
- Gloerfelt, Bailly, Juve (JSV 266:119-146, 2003) - subsonic cavity
  - Use DRP to discretize equations
  - Use non-reflecting boundary conditions + absorbing layer
  - Couple to an integral approach



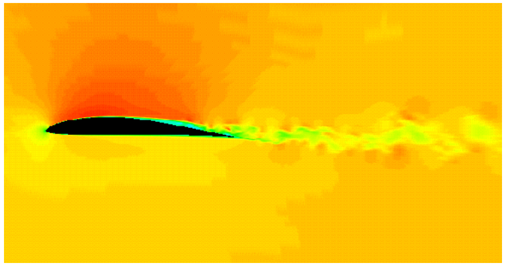
- Colonius, Freund, Lele (AIAA J, 38:2023, 2000) - supersonic jet
  - Use Pade methods for discretization
  - Use non-reflecting boundary conditions

LES examples

- Bogey, Bailly, Juve (Theor. Comp. Fluid Dyn, 16:273-297,2003) - jet
  - Use DRP to discretize equations
  - Use non-reflecting boundary conditions



- Uzun, Lynrinztis, Blaisdell (AIAA Paper No. 2004-0517) - jet
  - Use DRP to discretize equations
  - Use non-reflecting boundary conditions
  - Couple to an integral approach
- Sheen, Meecham (ASME Fluids Div, Sum. Mtg, 2:651-657, 2003) - jet
  - Coupled to an integral approach
- Oberai et.al. (AIAA J., 40(11):2206-2216, 2002) - incompressible airfoil
  - Use finite element incompressible LES
  - Use non-reflecting boundary conditions
  - Coupled to an integral approach



Euler, examples

Main applications found in literature -- rotor type simulations where flow disturbance is periodic and dominant in the creation of sound.

- Lee, et al (JSV 207(4):453-464, 1997) - rotor noise
  - Coupled to integral approach
- Lockard, Morris (AIAA J 36(6):907-914, 1998) - airfoil/gust
  - allowed for viscosity in some calcs
- Hixon (AIAA Paper No. 2003-3205), **Golubev** - cascade,airfoil/gust
  - Different approach, no time marching, space time mapping

LEE, examples

- Florea, Hall (AIAA J, 39(6):1047-1056,2001) - cascade/gust
  - Low-order discretization, finite volume
- Bailly, Juve (AIAA J, 38(1):22-29,2000) - apps.
  - DRP scheme
- Longatte, et al (AIAA J, 38(3):389-394,2000) - sheared ducted flow
- Lim, et al (JSV, 268(2):385-401,2003) - diffraction from impedance barriers
  - High order discretization
- Ozyorok, et al (JSV, 270(4-5):933-950,2004) - turbofan noise
- Chen, et al (JSV, 270(3):573-586,2004) - sound from unflanged duct
- Mankbadi et al (AIAA J, 36(2):140-147, 1998) - jet

Splitting (LEE-based) (Atassi, Grzedzinski, JFM 209:385-403)

This new unsteady velocity splitting appears as

$$\mathbf{u}' = \frac{1}{2c_p} s'(\mathbf{X} - \hat{i}U_\infty t)\overline{\mathbf{u}} + \mathbf{u}^{(R)} + \nabla\phi^*$$

The vortical part of the velocity still satisfies

$$\frac{D_0\mathbf{u}^{(R)}}{Dt} + \mathbf{u}^{(R)} \cdot \nabla\overline{\mathbf{u}} = 0$$

exactly as it did in the original splitting, but the boundary conditions are now defined so that there is no singularity

$$\mathbf{u}^{(R)} \cdot \mathbf{n} = 0 \qquad \mathbf{u}^{(R)} \cdot \tau = 0$$

The potential function is now governed by

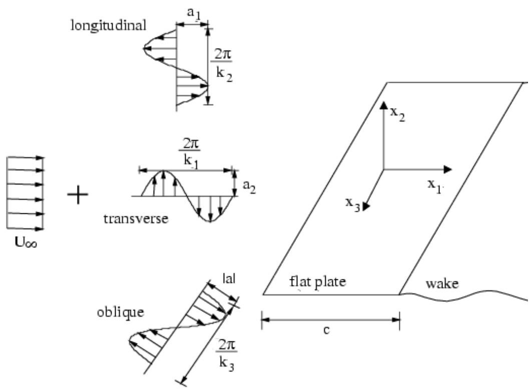
$$\frac{D_0}{Dt}\left(\frac{1}{\bar{c}^2}\frac{D_0\phi^*}{Dt}\right) - \frac{1}{\bar{\rho}}\nabla\cdot(\bar{\rho}\nabla\phi^*) = \frac{1}{\bar{\rho}}\nabla\cdot(\bar{\rho}\mathbf{u}^{(R)}) - \frac{1}{2c_p}\frac{\partial s'_\infty}{\partial t}$$

- the boundary condition along the surface is $\nabla\phi^* \cdot \mathbf{n} = 0$
- the jump of the potential velocity in a wake must be 0
- far upstream: $\nabla\phi^* \to \mathbf{u}'_\infty - \mathbf{u}^{(R)} - \frac{1}{2c_p}s'_\infty\overline{\mathbf{u}}_\infty$

Splitting, examples

✓ The vortical part is first solved analytically or numerically, and then the potential part is found numerically.
✓ Most often these problems are computed in the **frequency domain.**
✓ As shown on last slide, valid for subsonic, nonswirling flows

- Scott, Patrick/Grace, Atassi, (JCP 119(1):75-93, 1995, AIAA J. 31(1):12-19, 1993) - airfoil/gust
  - Low order-finite difference
  - Coupled to an integral approach
- Fang, Atassi, (JFE 115:573-579, 1993) - cascade/gust
  - Low order-finite difference
  - Novel non-reflecting boundary conditions
- Verdon, Hall (AIAA J. 29(9):1463-1471, 1991)- cascade/gust
- Peake et al. - (JFM 463(25):25-52, 2002) cascade/gust
- Golubev, Atassi (AIAA J 38(7):1142-1158,2000) - cascades/swirling flow
  - Vortical velocity is no longer the solution to a *homogeneous* equation



Summary

- We’ve consider the main features of computational aeroacoustic methods
    - Governing equations - hierarchy of approximations
    - Discretization schemes
        - DRP, Padé--spatial, LDDRK--time, methods for damping not described
    - Boundary conditions
    - Acoustic propagation methods for coupling near/far fields

- Many choices are problem dependent -- makes it difficult to incorporate good acoustic calculations in general CFD type codes

- If one is implementing these methods, it is good to use the CAA benchmark problems as preliminary method checks.

Thanks: Atassi, Tam, Bogey/Bailly, AME Dept., NCAD, Sondak

The End

More questions?

LES

Two - dimensional governing equations in conservative form

Cartesian co-ordinate system for a perfect gas

Spatially filtered (overbar), Favre (or density weighted) average (tilde)

Smagorinksy turbulence model

$$\tilde{u}_i = \frac{\overline{\rho u_i}}{\overline{\rho}}$$

$$\tilde{e}_T = \frac{\overline{\rho e_T}}{\overline{\rho}}$$

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{E}_e}{\partial x_1} + \frac{\partial \mathbf{F}_e}{\partial x_2} - \frac{\partial \mathbf{E}_v}{\partial x_1} - \frac{\partial \mathbf{F}_v}{\partial x_2} = \mathbf{0}$$

- ρ - density
- p - pressure
- **u** - velocity vector
- e_T - total energy
- τ - viscous stress tensor
- T - temperature
- R - gas constant
- γ - ratio of specific heats
- **q** - thermal conduction
- μ_t - turbulent viscosity
- $\mathcal{T}$- subgrid scale stress tensor
- C_s - Smagorinsky constant

$$\mathbf{U} = \begin{bmatrix} \overline{\rho} \\ \overline{\rho u_1} \\ \overline{\rho u_2} \\ \overline{\rho e_T} \end{bmatrix} \quad \mathbf{E}_e = \begin{bmatrix} \overline{\rho}\, \tilde{u}_1 \\ \overline{p} + \overline{\rho}\, \tilde{u}_1^{\,2} \\ \overline{\rho}\, \tilde{u}_1 \tilde{u}_2 \\ (\overline{\rho}\, \tilde{e}_T + \overline{p})\, \tilde{u}_1 \end{bmatrix} \quad \mathbf{F}_e = \begin{bmatrix} \overline{\rho}\, \tilde{u}_2 \\ \overline{\rho}\, \tilde{u}_1 \tilde{u}_2 \\ \overline{p} + \overline{\rho}\, \tilde{u}_2^{\,2} \\ (\overline{\rho}\, \tilde{e}_T + \overline{p})\, \tilde{u}_2 \end{bmatrix}$$

$$\mathbf{E}_v = \begin{bmatrix} 0 \\ \tilde{\tau}_{11} + \mathcal{T}_{11} \\ \tilde{\tau}_{12} + \mathcal{T}_{12} \\ \tilde{u}_i\, (\tilde{\tau}_{1i} + \mathcal{T}_{1i}) - q_1 \end{bmatrix} \mathbf{F}_v = \begin{bmatrix} 0 \\ \tilde{\tau}_{21} + \mathcal{T}_{21} \\ \tilde{\tau}_{22} + \mathcal{T}_{22} \\ \tilde{u}_i\, (\tilde{\tau}_{2i} + \mathcal{T}_{2i}) - q_2 \end{bmatrix}$$

$$\mathcal{T}_{ij} = \tilde{\tau}_{ij}\big|_{\mu_t} - \frac{2}{3}\overline{\rho} k_{sgs} \delta_{ij}$$

$$\mu_t = \overline{\rho}\left(C_s \sqrt[3]{\Delta x_1 \Delta x_2 \Delta x_3}\right)^2 \sqrt{\frac{1}{2\mu^2}\, \tilde{\tau}_{ij} \tilde{\tau}_{ij}}$$

DES

- Bisseseur et. al. (Aerospace Science Mtg. Proc. pg. 1673-1685, 2004)
  - Use high-order compact difference scheme
  - Couple to an integral approach

CFD, examples

Turbulence is modelled, usually high level of dispersion in discretization

- Kim, et al (AIAA Pap. 2003-3202) - general apps
  - FLUENT for near-field
  - Coupled to an integral approach
- Hendriana, et al (AIAA Pap. 2003-01-1361) - sideview mirror
  - FLUENT for near-field
  - Coupled to an integral approach
- Grace, Curtis (ASME, IMECE, NCAD 26:103-108, 1999) - cavity
  - FLUENT (URANS) for near-field
  - Coupled to solution of wave equation

Splitting Technique (Goldstein, JFM 89(3):433-468)

Ideally (uniform mean flow) the unsteady velocity can be split into solenoidal (vortical/entropic) and irrotational (potential) parts with separate governing equations. The components are linked through boundary conditions.

$$\nabla \cdot \mathbf{u}'_{(sol)} = 0 \qquad \nabla \times \mathbf{u}'_{(irrot)} = \mathbf{0}$$

For realistic flows (no shocks or swirl), the velocity components cannot be split as such.

- Potential governed by a single inhomogeneous, non-constant coefficients, convective wave equation forced by the solenoidal component
- Vortical part governed by homogeneous, non-constant coeffeicient, convective wave equation
- Entropic part governed by energy equation.

Used when the disturbance is vortical or entropic not acoustic.

Upstream where flow is uniform: $\mathbf{u} = U_\infty \hat{i} + \mathbf{u}'_\infty(x - U_\infty t, y, z) \qquad \nabla \cdot \mathbf{u}'_\infty = 0$

$$S = s'_\infty(x - U_\infty t, y, z) \qquad p \to p_\infty = constant$$

So entropy and velocity upstream are the boundary conditions imposed on the flow.

Splitting cont.

Equations for the split variables are derived from the nonconservative form of the governing equations with the energy equation expressed in terms of entropy.

$$\frac{D_0\rho'}{Dt} + \rho' \bigtriangledown \cdot \overline{\mathbf{u}} + \bigtriangledown \cdot (\overline{\rho}\mathbf{u}') = 0$$

$$\overline{\rho}\left(\frac{D_0\mathbf{u}'}{Dt} + \mathbf{u}' \cdot \bigtriangledown \overline{\mathbf{u}}\right) + \rho' \overline{\mathbf{u}} \cdot \bigtriangledown \overline{\mathbf{u}} = - \bigtriangledown p'$$

$$\frac{D_0}{Dt} = \frac{\partial}{\partial t} + \overline{\mathbf{u}} \cdot \bigtriangledown$$

$$\frac{D_0 s'}{Dt} = 0$$

One can show that the solution to this set of equations can be written as:

$$\mathbf{u}' = \bigtriangledown\phi + \mathbf{u}^{(I)}$$

$$p' = -\overline{\rho}\frac{D_0\phi}{Dt}$$

where

$$u_i^{(I)} = \frac{1}{2c_p} s'(\mathbf{X} - \hat{i}U_\infty t)\overline{u}_i + \mathbf{A}(\mathbf{X} - \hat{i}U_\infty t) \cdot \frac{\partial \mathbf{X}}{\partial x_i}$$

The components of the argument are Lagrangian coordinates of the steady flow fluid particles. The components of **X** are defined as

$X_2(x_1, x_2, x_3)$ $X_3(x_1, x_2, x_3)$ independent integrals of $\frac{dx_1}{\overline{u}_1} = \frac{dx_2}{\overline{u}_2} = \frac{dx_2}{\overline{u}_2} = dt$

$$X_1(x_1, x_2, x_3) = U_\infty \bigtriangleup (x_1, x_2, x_3)$$

$$\bigtriangleup(x_1, x_2, x_3) = \frac{x_1}{U_\infty} + \int_{-\infty}^{x_1} \left(\frac{1}{\overline{u}_1(\xi, y_s(\xi, X_2, X_3), z_s(\xi, X_2, X_3))} - \frac{1}{U_\infty}\right) d$$

Lighthill drift function, time for particles to travel along a streamline

Splitting cont.

Solenoidal part of flow is defined and then the single equation:

$$\frac{D_0}{Dt}\left(\frac{1}{\bar{c}^2}\frac{D_0\phi}{Dt}\right) - \frac{1}{\bar{\rho}}\bigtriangledown\cdot(\bar{\rho}\bigtriangledown\phi) = \frac{1}{\bar{\rho}}\bigtriangledown\cdot(\bar{\rho}\mathbf{u}^I)$$

must be solved, subject to the boundary condition on the surface: $\mathbf{n}\cdot\bigtriangledown\phi = -\mathbf{n}\cdot\mathbf{u}^{(I)}$

and far upstream: $\phi(\mathbf{x},t)\to 0, \quad x_1\to-\infty$

This splitting above leads to singular behavior of the solenoidal part along the solid boundary.

Time discretization

Consider the time discretization:

$$\mathbf{U}^{(n+1)} - \mathbf{U}^{(n)} \simeq \Delta t \sum_{j=0}^{3} b_j \left(\frac{d\mathbf{U}}{dt} \right)^{(n-j)}$$

$$\mathbf{U}(t + \Delta t) - \mathbf{U}(t) \simeq \Delta t \sum_{j=0}^{3} b_j \frac{d}{dt} \mathbf{U}(t - j \triangle t)$$

3 b's chosen so that the Taylor's series are satisfied to 3rd order. Leaving one free parameter b_0 to minimize dispersion error.

$$\boxed{\begin{aligned} b_0 &= 2.30255809 \\ b_1 &= -2.49100760 \\ b_2 &= 1.57434093 \\ b_3 &= -0.38589142 \end{aligned}}$$

Transform the discretized equation

$$(e^{-i\omega\Delta t} - 1)\hat{\mathbf{U}} \simeq \Delta t \sum_{j=0}^{3} b_j e^{ij\omega\Delta t}(-i\omega\hat{\mathbf{U}})$$

$$\boxed{\tilde{\omega} = \frac{i(e^{-i\omega\Delta t} - 1)}{\Delta t \sum_{j=0}^{3} b_j e^{ij\omega\Delta t}}}$$

b_0 chosen to minimize E_1. σ allows one to adjust the degree of emphasis placed on wave propagation (real part) or damping characteristics (imaginary part). Tam uses a value of 0.36

$$E_t = \int_{-0.5}^{0.5} \{\sigma[Re(\tilde{\omega} \Delta t - \omega \triangle t)]^2 + (1 - \sigma)[Im(\tilde{\omega} - \omega \triangle t)]^2\} d(\omega \triangle t)$$

Time cont.

Spurious numerical solutions exist because $\tilde{\omega}$ is not unique based on ω.

Optimization range has been selected based on the behavior of the approximated frequency. In particular, $\tilde{\omega}\Delta t$ is well behaved for values less than 0.6, and the optimization ranges from -0.5 to 0.5. For stability, the entire computation must be restricted to the range of $\omega\Delta t$ from -0.6 to 0.6.

Time step must be chosen to ensure numerical stability:

$$\Delta t < \frac{0.4}{1.75[M + \sqrt{1 + (\Delta x / \Delta y)^2}]} \frac{\Delta x}{c_0}$$

M is the mean flow Mach number, c_0 is the mean flow speed of sound, 0.4 is the value under which all of the roots of ω are damped.

Numerical damping due to the small imaginary part of the approximate frequency. This gives a more stringent requirement on the numerator above. (Details in Tam)

What we can learn from the far-field form

For low Mach number, M << 1 (Crow) $T_{ij} \approx \rho v_i v_j$

If the source is oscillating at a given frequency $\omega = 2\pi f$

The far-field approx to the source in Lighthill's equation can be written as

$$\rho v_i v_j \left(\mathbf{y}, t - \frac{|\mathbf{x}|}{c_0} + \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}| c_0} \right) = \rho v_i v_j (\mathbf{y}) e^{i\omega(t - |\mathbf{x}|/c_0 + \mathbf{x}\cdot\mathbf{y}/c_0|\mathbf{x}|)}$$

Therefore the solution becomes $(p - p_0)(\mathbf{x}, t) = \dfrac{\pi f^2 x_i x_j}{c_0^2 |\mathbf{x}|^3} \displaystyle\int_{-\infty}^{\infty} [\rho v_i v_j] d^3\mathbf{y}$

Scalings: velocity -- U, length -- L, f of disturbance -- U/L

$\lambda = \dfrac{c_0}{f} = \dfrac{c_0 L}{U} = \dfrac{L}{M}$ Acoustic wavelength/source length >> 1

Acoustic field pressure

$$p' = p - p_0 \sim \frac{U^2}{L^2} \frac{1}{c_0^2 |\mathbf{x}|} \rho_0 U^2 L^3$$
$$= \rho_0 M^2 U^2 \frac{L}{|\mathbf{x}|}$$

fourth power of velocity

Acoustic Power

$$P \sim |\mathbf{x}|^2 \frac{p'^2}{\rho_0 c_0} \sim \rho_0 L^2 U^3 M^5$$

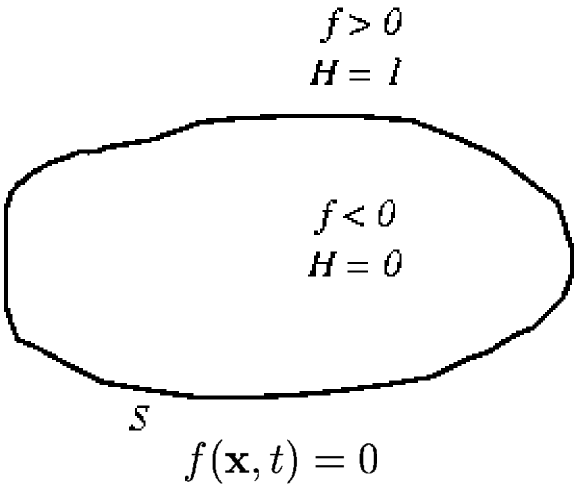
eighth power of velocity

FWH Eq.

Need a more general solution when:

- Turbulence is moving
- Two distinct regions of fluid flow
- Solid boundaries in the flow

- Define a surface S by *f = 0* that encloses sources and boundaries (or separates regions of interest)



- Surface moves with velocity **V**
- Heavy side function of *f* : *H(f)*
- Rule:

$$\frac{\partial H(f)}{\partial t} = \frac{\partial H}{\partial f}\frac{\partial f}{\partial t} = -V_i\frac{\partial H}{\partial f}\frac{\partial f}{\partial x_i} = -V_i\frac{\partial H}{\partial x_i} = -V_i\delta(f)\frac{\partial f}{\partial x_i}$$

Curle Eq.

When the surface is stationary the equation reduces to

Curle’s Equation

$$
\begin{aligned}
(\rho-\rho_0)H(f) \quad = \quad & \frac{\partial^2}{\partial x_i \partial x_j}\int_{\mathcal{V}} [T_{ij}] \frac{d^3\mathbf{y}}{4\pi c_0^2|\mathbf{x}-\mathbf{y}|} \\
& -\frac{\partial}{\partial x_i}\int_S [\rho v_i v_j + (p-p_0)\delta_{ij} - \sigma_{ij}] \frac{dS_j(\mathbf{y})}{4\pi c_0^2|\mathbf{x}-\mathbf{y}|} \\
& +\frac{\partial}{\partial t}\int_S [\rho v_j] \frac{dS_j(\mathbf{y})}{4\pi c_0^2|\mathbf{x}-\mathbf{y}|}
\end{aligned}
$$

Comments on analogy

- Analogy is based on the fact that one never knows the fluctuating fluid flow very accurately

- Get the equivalent sources that give the same effect

- Insensitivity of the ear as a detector of sound obviates the need for highly accurate predictions

- Just use good flow estimates…

- Alternative wave operators that include some of the refraction etc. effects that can occur due to flow nonuniformity near the source have been derived: Phillips' eq. , Lilley's eq.

- Using these is getting close to the direct calculation of sound.

FWH cont.

- Multiply continuity and Navier Stokes equation by H
- Rearrange terms, add and subtract appropriate quantities
- Take the time derivative of the continuity and combine it with the divergence of the NS equation

Differential form of the FWH Eq.

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\left(c_0^2(\rho-\rho_0)H(f)\right)$$

$$= \frac{\partial^2}{\partial x_i \partial x_j}\left(T_{ij}H(f)\right) - \overbrace{\frac{\partial}{\partial x_i}\left(\left(\rho v_i (v_j - V_j) + (p - p_0)\delta_{ij} - \sigma_{ij}\right)\frac{\partial H}{\partial x_j}\right)}^{\text{Dipole type term}}$$

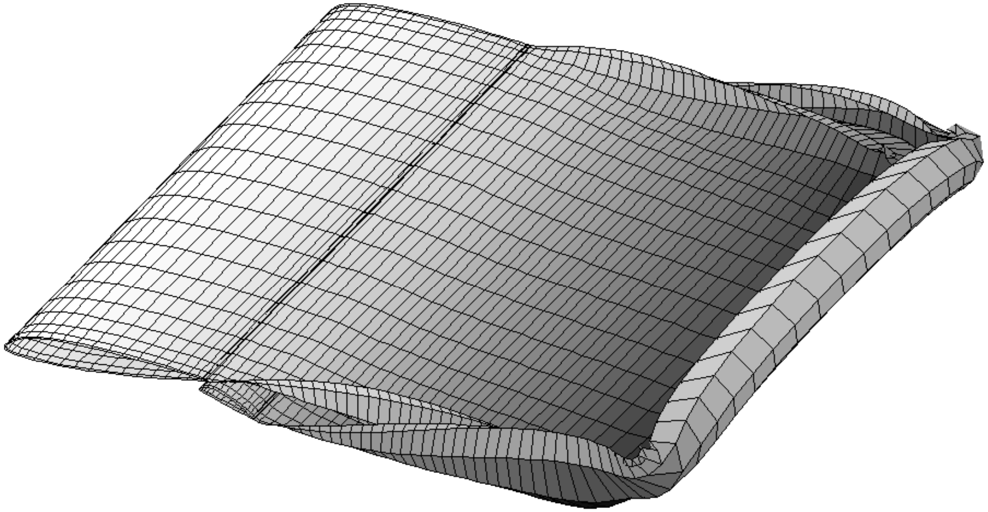
Quadrupole

$$+\underbrace{\frac{\partial}{\partial t}\left(\left(\rho(v_j - V_j) + \rho_0 V_j\right)\frac{\partial H}{\partial x_j}\right)}_{\text{Monopole type term}}$$

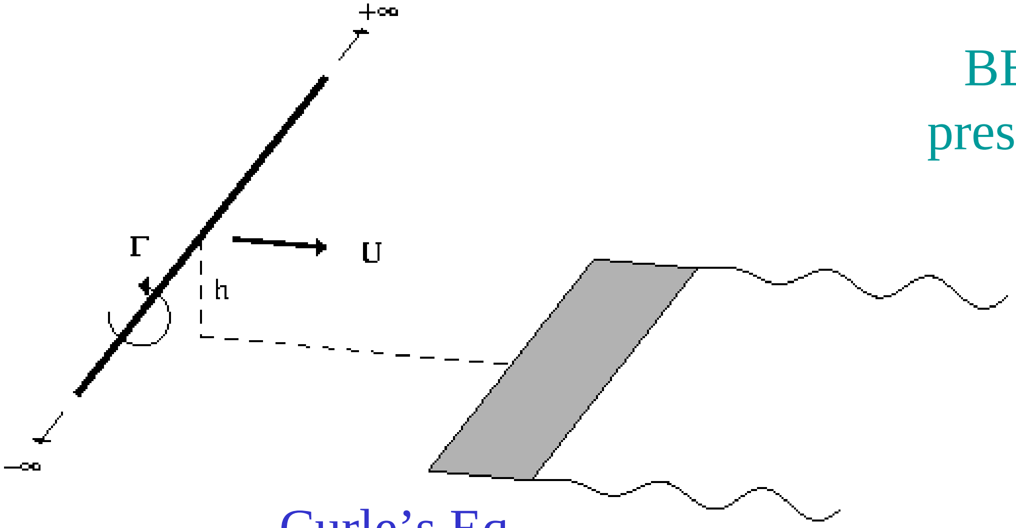
Example of application of FWH/Curle

We have a BEM for calculating the near field (surface forces)

(Wood, Grace)



3D BVI (rotor-type problem)



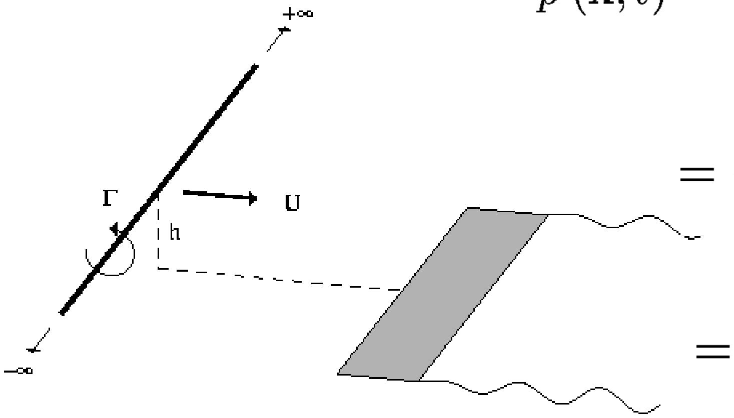
BEM computes unsteady pressure on the wing surface

Curle’s Eq.

$$\begin{aligned}(\rho-\rho_0)H(f) &= \frac{\partial^2}{\partial x_i \partial x_j}\cancel{\int_V [T_{ij}]}\frac{d^3\mathbf{y}}{4\pi c_0^2|\mathbf{x}-\mathbf{y}|} \\ &\quad -\frac{\partial}{\partial x_i}\int_S [\cancel{\rho v_i v_j} + (p-p_0)\delta_{ij} \cancel{-\sigma_{ij}}]\frac{dS_j(\mathbf{y})}{4\pi c_0^2|\mathbf{x}-\mathbf{y}|} \\ &\quad +\frac{\partial}{\partial t}\int_S \cancel{[\rho v_j]}\frac{dS_j(\mathbf{y})}{4\pi c_0^2|\mathbf{x}-\mathbf{y}|}\end{aligned}$$

$$\boxed{p'(\mathbf{x},t) = -\frac{\partial}{\partial x_i}\int_S [(p-p_0)\delta_{ij} n_j]\frac{dS}{4\pi|\mathbf{x}-\mathbf{y}|}}$$

Ex. cont.



$$p'(\mathbf{x},t) = -\frac{\partial}{\partial x_i}\int_S [(p-p_0)\delta_{ij}n_j]\frac{dS}{4\pi|\mathbf{x}-\mathbf{y}|}$$

$$= -\frac{\partial}{\partial x_2}\frac{1}{4\pi|\mathbf{x}|}\int_S [p-p_0]dS$$

far-field expansion

$$= -\frac{\partial}{\partial x_2}\frac{[L]}{4\pi|\mathbf{x}|}$$

integration of pressure is lift

$$= \frac{x_2}{4\pi c_0|\mathbf{x}|^2}\frac{\partial[L]}{\partial t}$$

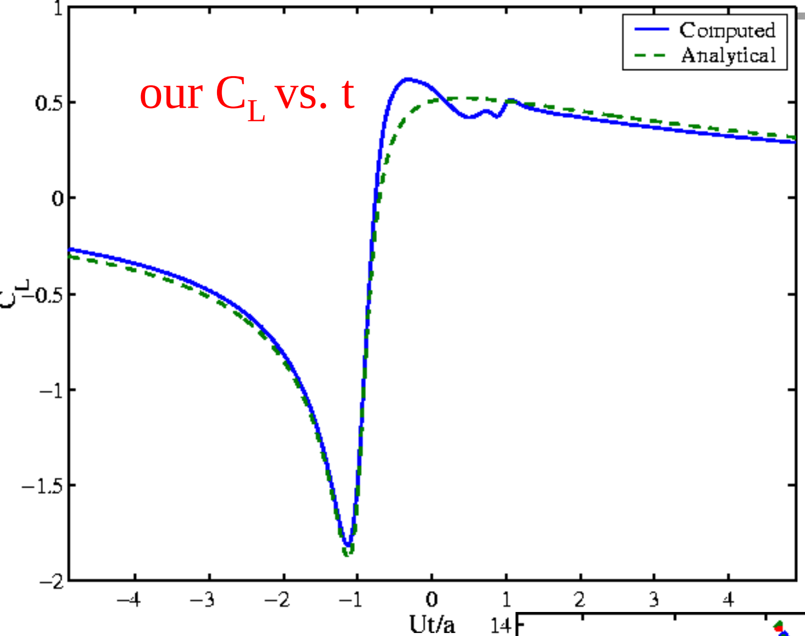
interchange space and time derivatives

Acoustic pressure in non-dimensional form

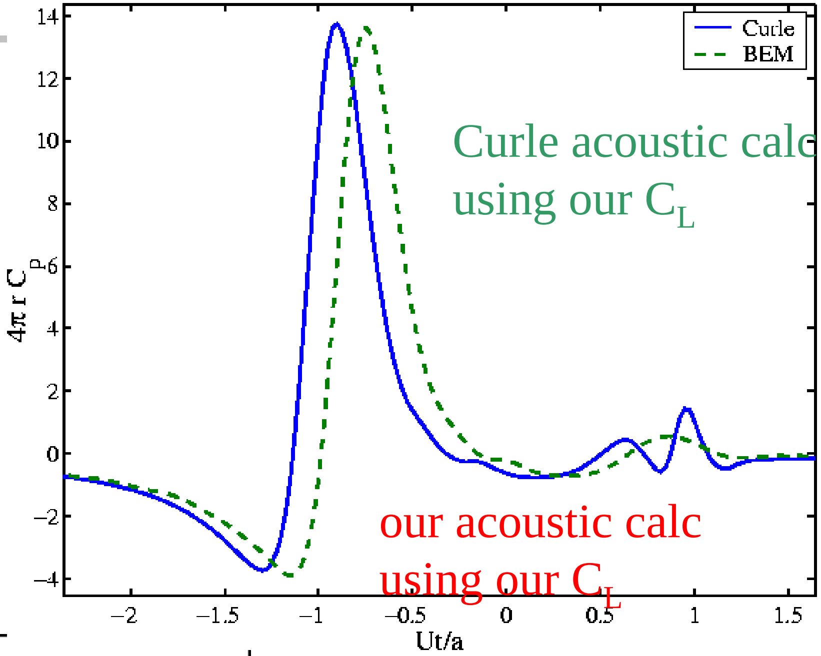
$$p' = \frac{M(AR)\cos\theta}{4\pi|\mathbf{x}|}\frac{\partial}{\partial t}[C_L]$$

$$\tau = t - M|\mathbf{x}|$$

Ex. cont.

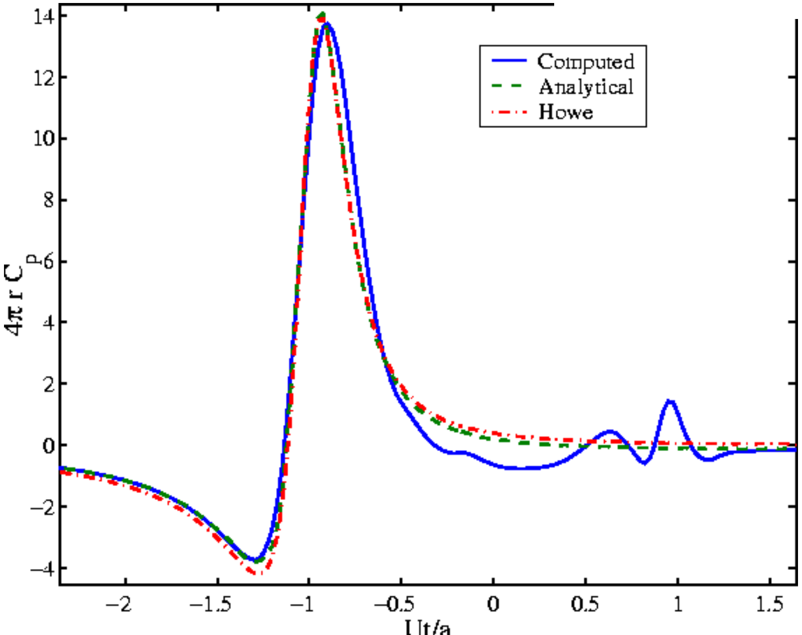


our C_L vs. t



Curle acoustic calc using our C_L

our acoustic calc using our C_L



Curle acoustic calc using our C_L

Purely analytic acoustic calc (based on analytic C_L

Additional info... FWH moving frame

FWH moving frame

Introduce new Lagrangian coordinate $\mathbf{y}(\boldsymbol{\xi},\tau) = \boldsymbol{\xi} + \int_{t_0}^{\tau} \mathbf{u}(\boldsymbol{\xi},\tau')d\tau', \tau > t_0.$

Inside integral, the δ function depends on τ now and

$$\int_{-\infty}^{\infty} f(y,\tau)\delta(g(\tau)) = \left[\frac{f(y,g)}{\frac{\partial g}{\partial \tau}}\right]$$

Where the additional factor that appears in the denominator is

$$\frac{\partial g}{\partial \tau} = -1 + \frac{1}{c_0}\frac{x_i - y_i}{|\mathbf{x}-\mathbf{y}|}\frac{dy_i}{d\tau} = -1 + \frac{u}{c_0}\cos\Theta = -1 + M\cos\Theta$$

angle between flow direction and **R**

because
$$\begin{cases} \mathbf{R} = \mathbf{x} - \mathbf{y}(\boldsymbol{\xi},\tau) \\ \dfrac{d\mathbf{y}}{d\tau} = \mathbf{u}(\boldsymbol{\xi},\tau) \\ \dfrac{x_i - y_i}{|\mathbf{x}-\mathbf{y}|} \end{cases}$$
= unit vector in the direction of **R**

FWH moving cont.

Volume element may change as moves through space

$$\frac{\rho}{\rho^*}d^3\mathbf{y} = d^3\boldsymbol{\xi}$$

density at τ = τ_0

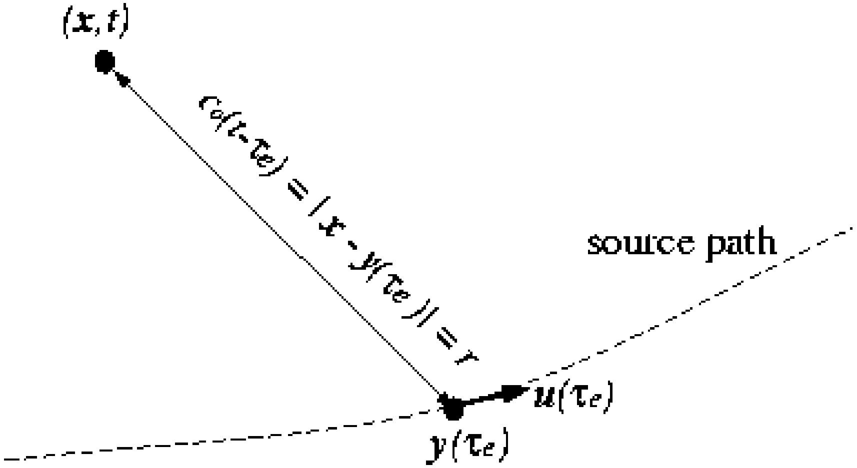
Volume element affected by Jacobian of the transformation

$$dS_i(\mathbf{y}) = n_i dS(\mathbf{y}) = n_i \frac{\rho^*}{\rho}\frac{|\partial f/\partial \mathbf{y}|}{|\partial f/\partial \boldsymbol{\xi}|} dS(\boldsymbol{\xi})$$

When control surface moves with the coordinate system … becomes ratio of the area elements of the surface S in the two spaces

Retarded time is calculated from

$$\tau - \tau_e = \frac{|\mathbf{x} - \mathbf{y}(\boldsymbol{\xi}, \tau_e)|}{c_0},$$



FWH moving cont.

When f is rigid: $u_j = V_j$

When body moves at speed of fluid: $V_j = v_j$

$$
\begin{aligned}
(\rho - \rho_0) H(f) \quad = \quad & \frac{\partial^2}{\partial x_i \partial x_j} \int_{\mathcal{V}} \left[\frac{T_{ij} \quad (\rho^* / \rho)}{4\pi c_0^2 R |1 - M \cos \Theta|} \right] d^3 \boldsymbol{\xi} \\
& - \frac{\partial}{\partial x_i} \int_S \left[\frac{(p - p_0)\delta_{ij} - \sigma_{ij} \xi_j}{4\pi c_0^2 R |1 - M \cos \Theta|} \right] dS(\boldsymbol{\xi}) \\
& + \int_S \left[\frac{\rho v_n}{4\pi c_0^2 R |1 - M \cos \Theta|} \right] dS(\boldsymbol{\xi})
\end{aligned}
$$

Square brackets indicate evaluation at the retarded time τ_e

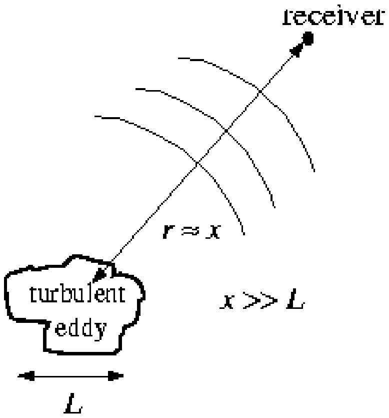
Doppler shift $\dfrac{1}{1 - M \cos \Theta}$

> 1 for approaching subsonic source

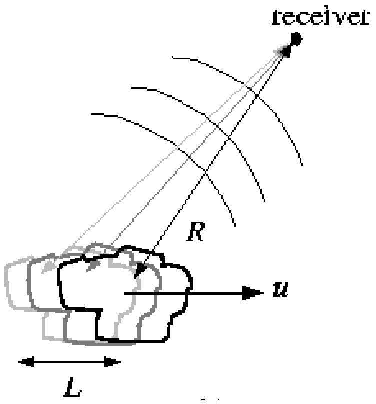
< 1 for receding subsonic source

accounts for frequency shift heard when vehicles pass

Turbulent noise sources

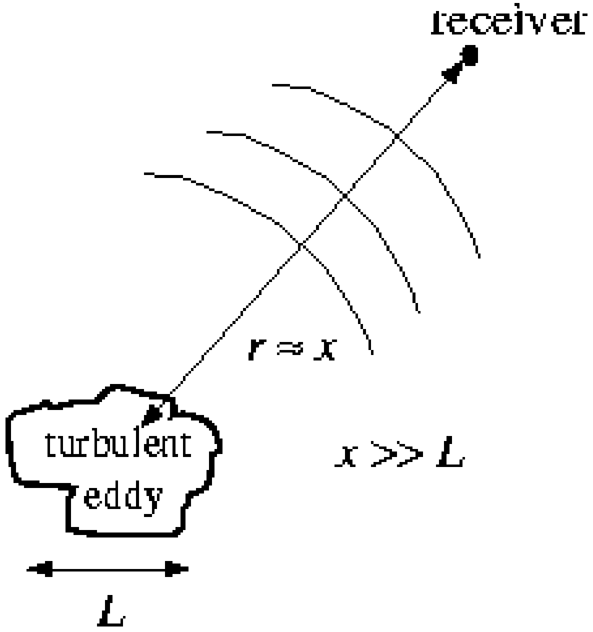


Stationary turbulence (low M)



Moving turbulence (high M)

Stationary turbulence (Low M)

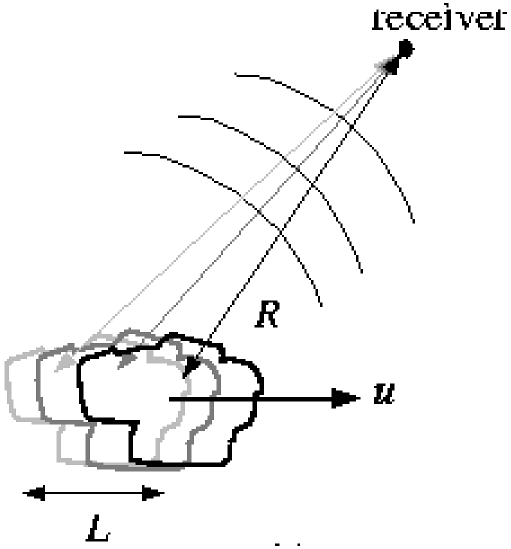


Far-field form

$$(p - p_0)(\mathbf{x}, t) = \frac{x_i x_j}{4\pi c_0^2 |\mathbf{x}|^3} \frac{\partial^2}{\partial t^2} \int [\rho_0 v_i v_j] d^3\mathbf{y}$$

From before…. Pressure goes as fourth power of velocity and power as eighth power of velocity

Moving turbulence



$$(p - p_0)(\mathbf{x}, t) = \frac{1}{4\pi c_0^2}\int \left[\rho^* \frac{\partial}{\partial t}\left\{\frac{R_i}{R(1-M\cos\Theta)}\frac{\partial}{\partial t}\left(\frac{R_j v_i v_j}{R(1-M\cos\Theta)}\right)\right\}\frac{d^3\boldsymbol{\xi}}{R(1-M\cos\Theta)}\right]$$

Pressure goes as **scaled** fourth power of velocity

$$\frac{\rho^*}{c_0^2}\frac{U^2}{L^2}\frac{1}{R}U^2L^3\frac{1}{(1-M\cos\Theta)^3} = \frac{L}{R}\frac{1}{(1-M\cos\Theta)^3}\rho^* U^2 M^2$$

Power goes as eighth power **scaled** by

$$\frac{1}{(1-M\cos\Theta)^6}$$

Example

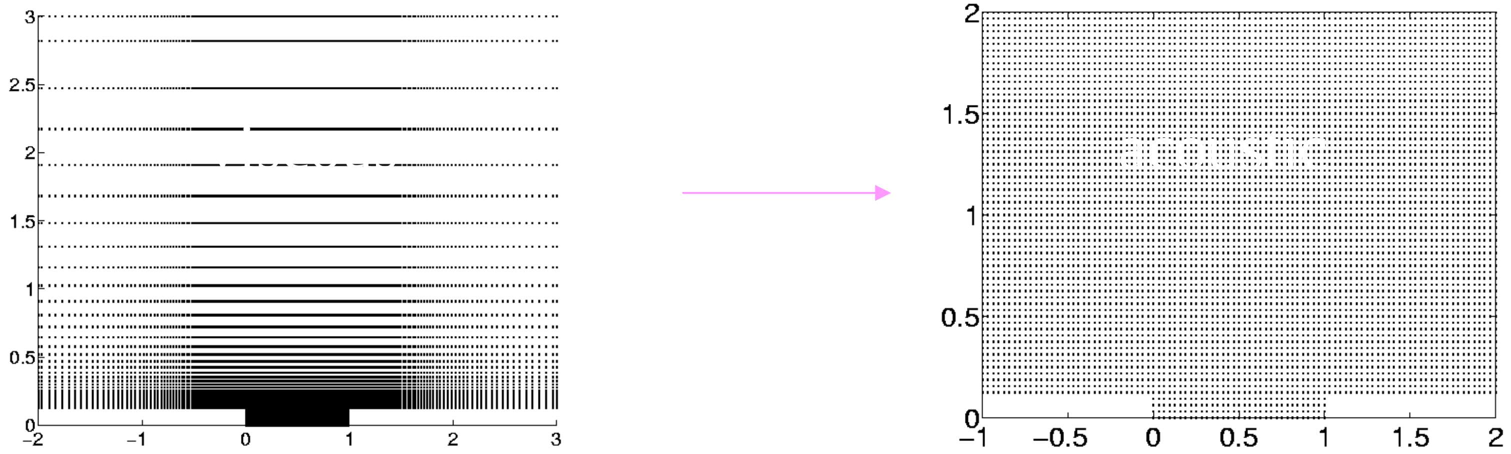
Method based on :

Unsteady CFD -> forced wave equation solved numerically

$$\left(\frac{1}{c_\infty^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)p=\rho_0\nabla\cdot(\vec{v}\cdot\nabla)\vec{v}$$

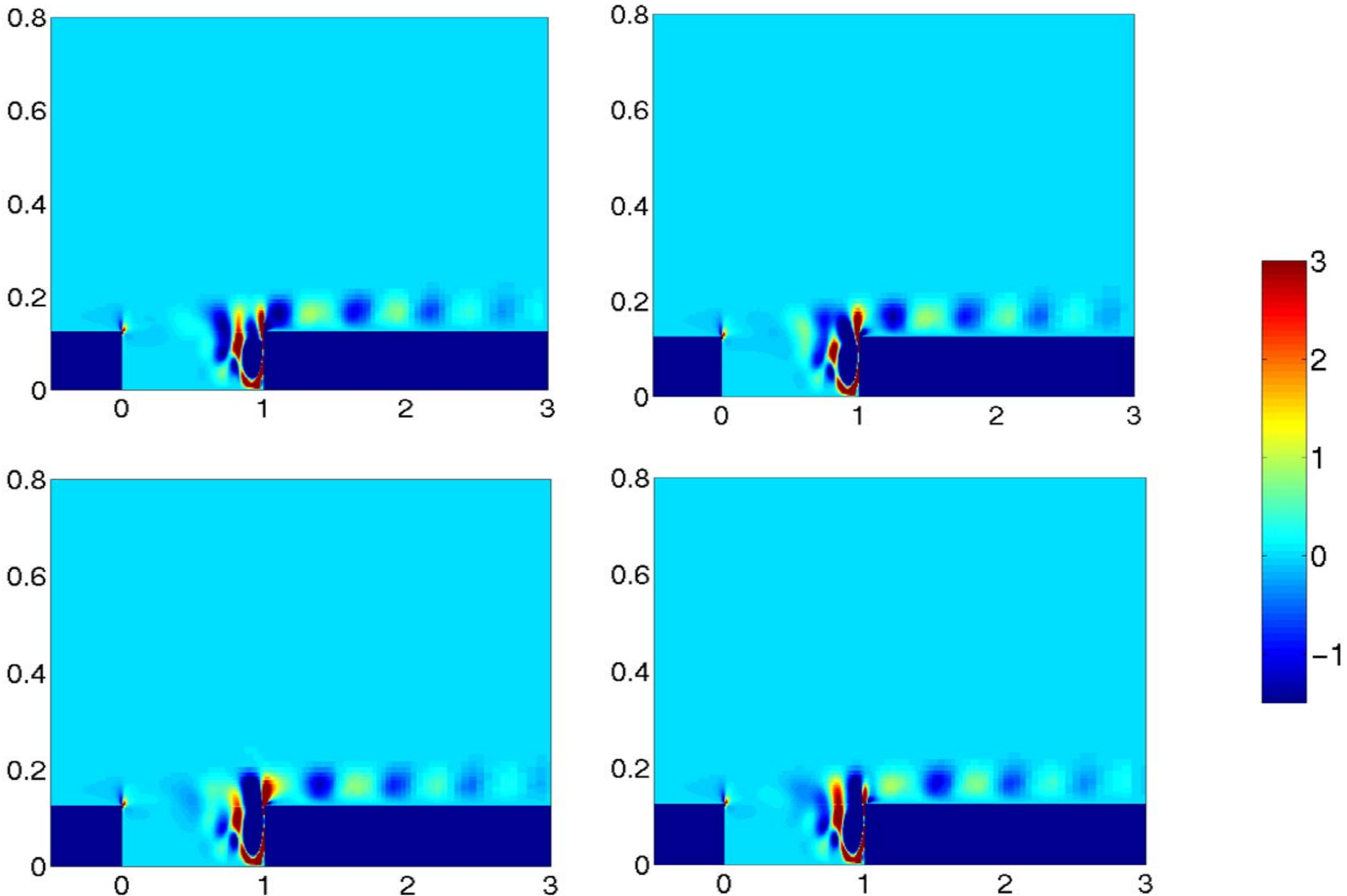
Goal :

Make use of existing CFD through a hybrid method for computational aeroacoustics (i.e. no integral formulation for the acoustics)

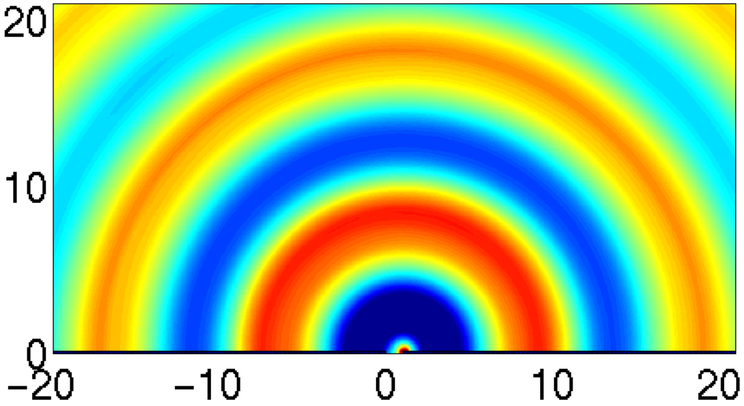


Results for U = 33.1m/s, L/D = 8.0, Re=8100

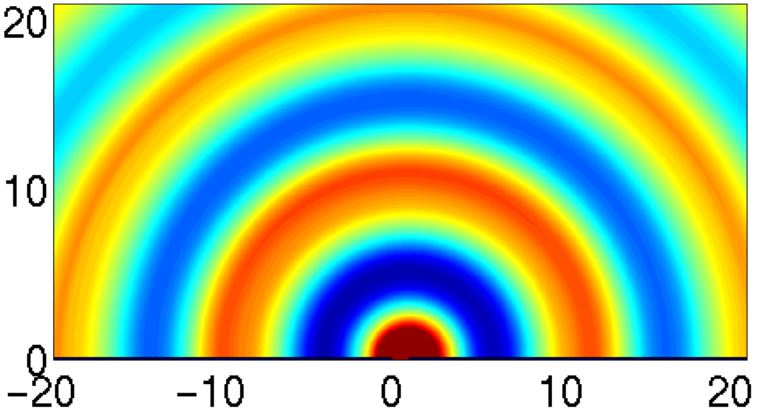
Unsteady non-dimensional source term



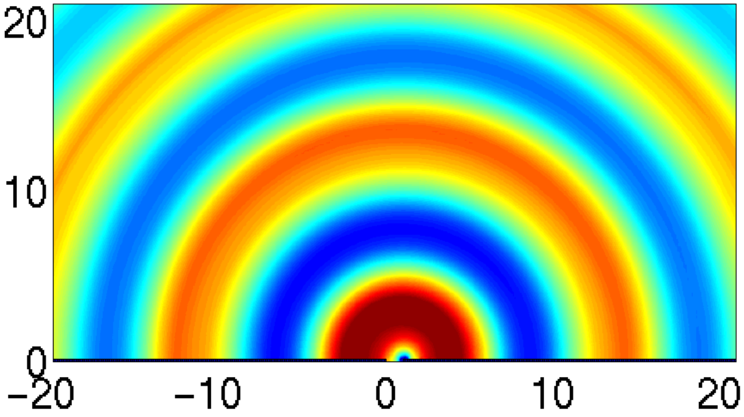
Acoustic pressure field



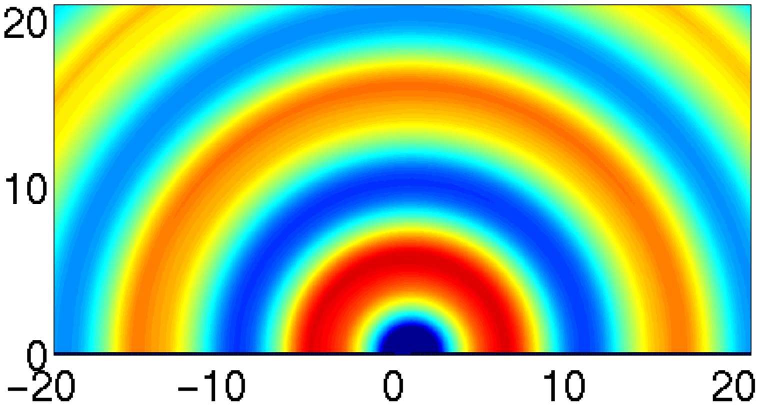
t=0.5168 s



t=0.5247 s



t=0.5326 s



t=0.5405 s